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ON BOUNDED LINEAR MAPPINGS OF SPECTRUM IN HILBERT SPACES

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Abstract
The decomposition of the spectrum of bounded linear mapping is a fundamental topic in functional analysis that has wide ranging implications in various fields. In this study the context of bounded linear mappings will be discussed. The spectrum of a linear mapping provides crucial information about its eigenvalues and eigenvectors, and the overall behaviour of the operator. By understanding the decomposition of spectrum, the study will gain great insights into different components that contribute to the overall content of a spectrum, leading to deeper understanding of the mapping's properties of a spectrum in a Hilbert space.

Key terms: Decomposition, Hilbert space, mappings, spectrum.

INTRODUCTION

The decomposition of spectrum is a fundamental concept in functional analysis, specifically in the study of operators on Banach spaces. The spectrum of a bounded linear operator provides valuable insights into its properties and behaviour. It covers the set of all complex numbers for which the operator fails to have an inverse. The notion of compact operator and its compactness,

Bounded Linear Mapping

A bounded linear mapping, also known as bounded linear operator or bounded linear transformation, is a fundamental concept in functional analysis. It is a mapping between two normed vector spaces that preserves both vector addition and scalar multiplication while also being bounded in terms of its operator norm.

Formally, let X and Y be normed vector spaces over the same field, either the real numbers or complex numbers. A linear mapping $T:X \geq Y$ is said to be bounded if there exists a constant $M \geq 0$ such that for all $x \in X$, the norm of the image of x under T, denoted by $||T(X)||$, satisfies $||T(X)|| \leq M||X||$, where $||x||$ and $||T(x)||$ denote the norms of x and T(x) in X and Y, respectively.

In simpler terms, a bounded linear mapping ensures that the ratio of the norm of the image of a vector to the norm of the vector itself is always bounded by a constant M. This property ensures that the mapping does not affect the norms of the vectors, and it allows for a well behaved and continuous transformations between normed vector spaces.

Examples of bounded linear mappings include *matrix multiplication* (when X and Y are finite-dimensional vector spaces), *differentiation operators, integral operators* and many other linear transformations encountered in mathematical analysis and various applications.

Definition 1

Normed Vector Space: A normed vector space is a vector space equipped with a norm, which is a function which assigns a non-negative length or size to each vector in the space. The norm satisfies certain properties such as non-negativity, scalability and the triangular inequality.

Linear Mapping: A linear mapping, also known as linear operator or linear transformation, is a function between two vector spaces that preserves vector addition and scalar multiplication. In other words, for vector spaces X and Y, a function $T:X \geq Y$ is linear if it satisfies $T(X+Y) = T(X) + T(Y)$ and $T(ax) = aT(x)$ for all x, y in X and scalar a.

Spectrum of a Linear Map: Bounded Linear Mapping: A linear map, also known as a linear operator or linear transformation is a function between vector spaces that preserves vector addition and scalar multiplication. If X and Y are vector spaces, a linear map $T:X \geq Y$ satisfies $T(x+y) = T(X)+T(Y)$ and $T(ax) = aT(x)$ for all vectors x,y in X and scalar a.

Eigenvalue: Given a linear map T:x ≥X, an eigenvalue of T is a scalar λ such that there exists a non-zero vector x in X, called an eigenvector satisfying T(x) = λx.

Spectrum: The spectrum of a linear map refers to all complex numbers λ for which the equation T(x) = λx has non-zero solution x in the underlying vector space X. Normally denoted as Q(T).

Hilbert Space

A Hilbert space is a complete inner product space. It is a vector equipped with an inner product, which is a bilinear, positive-definite, and Hermitian form that satisfies certain properties. Hilbert spaces provide a rich framework for studying various aspects of functional analysis, including linear operators, orthogonal projections and function spaces.

Inner Product: An inner product on a vector space x is a function that takes two vectors x and y in X and assigns a complex number (x, y) satisfying the following properties:

1) **Linearity in the first argument:** {αx + βy, z} = α{x, z} for all α,β in the underlying scalar field and all vectors x, y, z in x.
2) **Conjugate symmetry**: {y, x} * for all x, y in X, where * denotes complex conjugation.
3) **Positive definiteness**: {x, x} ≥ 0 for all x in X and {x, x} = 0 if and only if x is the zero vector.

Orthogonality: In a Hilbert space, two vectors x and y are said to be orthogonal if their inner product {x, y} is zero. Orthogonality plays a crucial role on Hilbert spaces, allowing for concepts such as orthogonal projections, and the decomposition of vectors into orthogonal components.

Orthonormal Basis: An orthonormal basis for Hilbert space is a collection of vectors that forms a basis (i.e linearly independent and spanning) and satisfies the additional property being orthogonal and having unit length.

Orthogonal Projection: An orthogonal projection is a linear operator in a Hilbert space that maps vectors onto subspace M of a Hilbert space, an orthogonal projection P⊥M is a linear operator that maps any vector x in the space onto closest vector in M.

Decomposition of Spectrum

The spectrum of a linear operator T that operates on Banach space x(a fundamental concept of a functional analysis, consist of all scalars of λ such that the operator λ -T does not have bounded inverse on x. The spectrum has standard decomposition into three parts:

A point spectrum: consists of the eigenvalues of λ for which there exists at least one non-zero vector v such that Av = λv. These eigenvalues are called the point eigenvalues, and the corresponding eigenvectors are called the point eigenvectors. The point spectrum is characterised by isolated eigenvalues with finite multiplicity. Each eigenvalue in the point spectrum is referred to as a point eigenvalue, and the associated eigenvectors are known as point eigenvectors. The eigenvectors in the point spectrum are non-zero and serve as a direction in the vector space that remains unchanged, apart from scaling, when acted upon by the linear operator or matrix. The point spectrum is characterised by isolated eigenvalues, which means each eigenvalue is distinct and separated from other. These isolated eigenvalues have finite multiplicity, indicating the number of linearly independent eigenvectors associated with each eigenvalue. The point spectrum is significant as it provides valuable information about the behaviour and the properties of the linear operator or matrix. The eigenvalues and eigenvectors in the point spectrum can reveal essential features such as stable modes or dominant components in the system represented by the operator or matrix.

A Continuous Spectrum: is a subset of the spectrum of a linear operator or matrix. It includes eigenvalues λ that are embedded in a continuous interval or subset of the complex plane. Unlike the point spectrum, the continuous spectrum does not have corresponding eigenvectors. For eigenvalues in the continuous spectrum, there are no individual eigen vectors that satisfy the equation $Av = \lambda v$. However, these eigenvalues are associated with a generalised notion of eigenvectors or eigenvectors of the adjoint operator. These generalised eigenvectors are not unique and do not fully capture the behaviour of the system. Instead, they provide a way to describe the behaviour of the operator or matrix when an eigenvalue lies in the continuous spectrum. The generalised eigenvectors span a subspace associated with each eigenvalue in the continuous spectrum. The continuous spectrum often arises in situations where the linear operator or matrix has an infinite-dimensional space or when there is a continuous distribution of eigenvalues.

A Residual Spectrum: It comprises all remaining eigen-values that are neither in the point spectrum nor in the continuous spectrum. These eigenvalues are often associated with infinite-dimensional eigenspaces and can have infinite multiplicity. The presence of eigenvalues in the residual spectrum often arises when dealing with linear operators or matrices acting on infinite-dimensional spaces, such as in functional analysis or partial differential equations. The residual spectrum is important in understanding the behaviour of the linear operator or matrix in situations where neither isolated eigen-values (point spectrum) nor eigen-values embedded in a continuous interval (continuous spectrum) are present. These eigen-values in the residual spectrum can have significant implications for the stability, convergence, and long-term behaviour of the system represented by the operator or matrix.

Spectral Theorem

The spectral theorem is a fundamental result that provides a general framework for the decomposition of a self-adjoint or normal operator, unitary operators and spectral theorem for compact self-adjoint operators. For self-adjoint operators on finite-dimensional spaces, the spectral theorem states that, every bounded self-adjoint operator on a Hilbert space has a complete set of orthonormal eigen-vectors and its spectrum is real and bounded. For normal operators on finite-dimensional or infinite-dimensional spaces, the spectral theorem states that such operators can be decomposed into direct integral (continuous sum) of their eigenspaces associated with their eigenvalues. The spectral theorem plays a crucial role in understanding the decomposition of the spectrum for self-adjoint operators, providing a basis for diagonalisation and spectral representations.

Theorem 1

Let $A \in M_{n*n}(\mathbb{R})$ be symmetric.

Then, 1) A has real eigen-value.
2) There exist a basis of $\mathbb{R}^n$ which consists of e-vectors of A.

I). Supposing $\lambda \in \mathbb{C}$ is an e-value, the study sought to show $\lambda = \bar{\lambda}$

$A\bar{V} = \bar{\lambda}\bar{v}$(eigen value equation).
$(AV)^T \bar{V} = (TV)^T \bar{V} = \lambda V^T V$ (1) which is also equivalent to
$V^T A^T \bar{V} = V^T A\bar{V} = V^T \lambda \bar{V} = V^T \bar{\lambda} V = \bar{\lambda} V^T \bar{V}$ (2)
A is symmetric
$\Rightarrow \lambda = \bar{\lambda}$ and so $\lambda \in \mathbb{R}$

$V^T V = |V_1|^2 + |V_2|^2 + \text{------} + |V_n|^2$
$\Leftrightarrow$For symmetric A,$\exists$ orthogonal matrix R such that R^{-1} AR is diagonal.

Proof

Suppose $\lambda_1 (\in \mathbb{R})$ is an eigen-value of A, it's going to have an eigen-vector V_1 normalize the vector so that $||v_1|| = 1$
Extend to a basis of $V_1, U_2, U_3, \text{- - - - - - -}, U_n$
Allows you to take a basis and turn it into a normal for any finite inner product space.
By Graham Schimidt we get an orthonormal basis $V_1\ V_2, \text{- - - - - -} V_n$
$P = [\underline{V}_1, \underline{V}_2, \underline{V}_n]$ P is orthogonal $P^{-1} = P^T$ P is orthogonal $P^{-1} = P^T$.
Let B= P^{-1} AP we want to show that this is diagonal.
Step 1: Show B is symmetric
$B = P^T AP$.
$B^T = P^T AP = B^T = (P^T(AP))^T = (AP)^T P = P^T A^T P$
$B^T = (P^T(AP))^T = (AP)^T P = P^T A^T P = P^T A P$
Step 2: Consider $B\underline{e}_1 = 1^{st}$ column of your matrix.

$$P^T A p\underline{e}_1 = p^T A\underline{V}_1 = P^T \lambda_1 \underline{V}_1 = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$P^T = \begin{bmatrix} \leftarrow & V_1 & \rightarrow \\ \leftarrow & V_2 & \rightarrow \\ \leftarrow & V_n & \rightarrow \end{bmatrix} \text{ Orthonormal.}$$

$$B = \begin{bmatrix} \lambda & 0\hat{\ }0 \\ 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$$

Proof by Induction

True up to n-1

∴ ∃ an orthogonal Q such that $Q^{-1}CQ = D$ which is diagonal.

$$\text{Define } R = P\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$$

$$1)\quad R^{1} = \begin{pmatrix} 1 & 0 & P^{-1} \\ 0 & q^{-1} & \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & Q^{T} \end{pmatrix} P^{T} = R^{T}$$

∴ R is orthogonal.

$$2).\ \text{Compute } R^{-1}AR = R^{T}AR = \begin{pmatrix} 1 & 0 \\ 0 & Q^{T} \end{pmatrix} P^{T}A \begin{pmatrix} 1 & 0 \\ 0Q & \end{pmatrix}$$

$$R^{-1}AR = R^{T}AR = \begin{pmatrix} 1 & 0 \\ 0 & Q^{T} \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & QC^{T}Q \end{pmatrix}$$

The study shows that it's diagonalised.

Spectral Theorem for Normal Operators

Normal operator S is one that commutes with its own adjoint S*.
Invertible normal operator G can be written as G=KP = PK where K is unitary and P is positive self-adjoint.
Unitary operators (respectively positive self-adjoint operator) has an orthonormal basis of eigenvectors.

Simultaneous Diagonalisation: Suppose A and B are commuting matrices. Then, for an eigenvector v of A we have,

ABv = BAv = Bλv = λBv

Where λ is the eigenvalue of A associated with eigenvector v. Thus if we denote by Vλ the space of all vectors v(including 0!) for which Av = λv, then B takes this subspace to itself. If A is diagonalisable, then we write basis of eigenvectors, hence we can express every vector v as a (unique) linear combination of elements $v_\lambda \in V_\lambda$ as we vary λ over eigenvalues of A. Further B takes V_λ to itself. In addition, B is diagonalisable then we can further write V_λ in terms of basis of eigenvectors for B. (This gives basis of our original vector space which consist of simultaneous eigenvectors for both A and B (since each non-zero element of V_λ is an eigenvector of A).

The Invertible Case: This is applied to the decomposition G = KP = PK for normal invertible matrix G which we saw above. Recalling that K is unitary and P is positive definite. By the above argument, we can find a unitary basis consisting of simultaneous eigenvectors for K and P. It's clear that these are eigenvectors for G as well. Let Vk,p denote the subspace of vectors v so that Kv= kv and Pv=pv; in other words, Vk,p is the subspace spanned by simultaneous eigenvectors for K and P with eigenvalues K and P respectively. Let V be in Vk,p and w be in Vk',p' with (k,p)≠(k'p'). We then calculate:

P v,w Pv,w = v,Pw = P'v,w

Where we have used the fact that P' is real. If P≠P', then this says <v,w> = 0. Similarly, <v,w>=>Uv,Uw> = kk^{-1}<v,w>.

Now k' is a complex number of norm 1 so kbar' its multiplicative inverse. Thus k≠k', then <v,w> = 0. It follows that Vk',p' is contained in $V\perp_{k,p}$ if (k,p) ≠ (k'p').

Let $E_{k'p'}$ = 0 if (k,p) ≠ (k',p'). Moreover

K = $\sum_k K \sum_p E_{k,p}$ and P = $\sum_p P \sum_K E_{k,p}$

Since both K and P are diagonalisable. It follows that G = KP = $\sum_{k,p} kpEk,p$

Now note that λ = kp is eigenvalue G. Moreover, since P is positive, we have |λ|=P, so λ= λ', iff (k,p)=(k'p'). In other words G is a sum of matrices of the form λE_λ where λ is an eigenvalue G and E_λ is an orthogonal projection onto the subspace where G acts as multiplication by λ(this is called the λ-eigenspaces of G). Moreover, E λE_u =0, if ≠ μ.

Now suppose that M is a normal operator, i.e M commutes with M*, on a finite dimensional complex inner-product space U.As seen earlier, V has an orthonormal basis. Let M also denote the matrix of M with respect to this bases as seen earlier the matrix of the adjoint operator is also M* with respect to this orthonormal basis. Consider the characteristic polynomial P(T)=det(M-T.1). This is a non-zero polynomial since it's leading coefficient T^n where n is the dimension of v. Hence there is an integer k so that p(k)≠0. It follows that G =M-K.1 is an invertible matrix. Clearly G*=M*-K.1 commutes with M and thus also with G. In other words, G is a normal invertible matrix. By what has been proved above, we have an expression G = $\sum_\lambda \lambda E_\lambda$ where E_λ are orthogonal eigenvectors each such vector is in the image of E_λ for some λ (note that G has only non-zero eigenvalues).

It follows that E =$\sum_\lambda E_\lambda$ is identity on this basis of eigenvectors; hence E = 1.We thus obtain identity M = G+K.1=$\sum_\lambda(\lambda+K)E_\lambda$. We have thus obtained spectral decomposition of the normal operator M.

Conversely, suppose M =$\sum_\lambda \lambda E_\lambda$ where E_λ are orthogonal idempotents with $E_\lambda E_u$ = 0. If λ ≠ μ. Then M* = $\sum_\lambda E_\lambda$, since E^*_λ = E_λ. Moreover, M commutes, with all E_λ and this commutes with M* as well.

Spectral Theorem for Self-adjoint Operators

Given a self-adjoint A∈L(H) and interval I is in I = [-||A|| - δ], 1_0 = [-||A||,||A||],take a non-zero V∈H and set

μ(f) = μA,v(f) = (f(A)v,v), f∈C …………..(1)

Let f ≥ 0 ⟹ f(A) f ≥ 0, hence μ(f) ≥ 0. Thus μ gives a positive radon measure on (1).

(f(A)v,v) =$\int_I f\, du$

Now we define

$W = W_{A,V}:C(I) \to H$

$W(f) = f(A)v$.

Note that, if also $g \in C(I)$,

$(W(f), w(g)H = (f(A)v, g(A)v)$

$= (g^{-}(A)\ f\ (A)\ v, v)$

$= \int_I fg^{-} du$

$= (f,g)L^2(I,u)$

Consequently, W has continous extension to $W:L^2(I,\mu) \to H$, an isometry.

Note that the range of W as shown above is the closure in H of

C(A,v) = Span

We call $\subset (A, v) \subset H$ the cyclic subspace of H generated by A and v. If C(A,v) = H, we say v is a cyclic vector for A.

The special case of the theorem

If $A \in L(H)$ is self adjoint and has cyclic vector v, then; $W:L^2(1,\mu) \to H$ is unitary and $W^{-1}Aw\ f(x) = xf(x)$, $\forall f \in L^2(I, \mu)$.

Proof

The unitary follows from $W:L^2(I, \mu) \to H$ to $C(A,v) = span\{v, A_V, A^2v, ...\}$. To get $w^{-1}\ AW\ f(x) = xf(x)$, $\forall f \in L^2(1,\mu)$, we should note that W(xf) = AW(f), the first identity by W(f)= f(A)w, with f replaced by xf, with $\|f(A)\| \le \sup |f|$, f $\ge 0 \Longrightarrow f(A) \ge 0$, $f(A)^*\ f^{-}(A)$. The identity W(xf)=Af(A)v = AW(f) holds first for $f \in C(I)$, hence by continuity, for all $f \in L^2(I,\mu)$.

We cannot say that a given self-adjoint $A \in LH$ has a cyclic vector, but we can assume H to be separable for simplicity.

Proposition

If H is separable and $A \in L(H)$ is self-adjoint, then $\exists\ V_j \in H$ such that $C(A,v_j)$ are mutually orthogonal subspaces of H, with the span dense in H.

Proof

Let $\{w_j : j \in \mathbb{N}\}$ be a dense subset of H, all $w_j \neq 0$. Take $v_1 = w_1$, and construct $C(A,v_1) = H_1$, as above. Recalling that $A:H_1 \to H_1$.

If $H_1 = H$, we are done. If not we proceed as follows:

We can claim that, whenever $H_1 C H$ is linear subspace,

$A:H_1 \to H_1 \Rightarrow A:H_1 \perp \to H_1 \perp$

In that, if $v \in H$, $w \in H_1 \perp$,then

$(u,Aw) = (Av,w) = 0$(given $Au \in H_1 \perp$), so follows $A:H_1 \to H_1 \Rightarrow A:H_1 \perp \to H_1 \perp$

To continue, we consider the 1st $j \ge 2$ such that $wj \notin H_1$, and let v_2 denote the orthogonal projection of w_j onto $H_1 \perp$.Then set $H_2 = C(A,v_2) \subset H_1 \perp$.Clearly $H_1 \oplus H_2$ contains span $\{wk : 1 \le k \le j\}$. If $H_1 \oplus H_2 = H$, we are done if not we get countable sequence of mutually orthogonal spaces $H_K = C(A,vk)$, whose span contains wj for all $j \in \mathbb{N}$ so is dense in H.

Spectral Theorem for Self-Adjoint Operators: The Spectral Theorem for Self-Adjoint Operators applies to a self-adjoint operator A on a Hilbert space H. It states that there exists a unique spectral measure E on Borel sets of the spectrum σ(A) of A, denoted by Borel (σ(A)), such that A can be decomposed as:
$A = \int \lambda \, dE(\lambda)$,
Where the integral is understood in the strong operator topology. The spectral measure E satisfies certain properties such as being a projection-valued measure.

Spectral Theorem for Normal Operators: The spectral Theorem for Normal Operators is a more general result that applies to a normal operator N on a Hilbert space H. It states that there exists a unique spectral measure E on the Borel sets of the spectrum (σ(N)), such that N can be decomposed as:
$N = \int \lambda \, De(\lambda)$,
Where the integral is understood in the strong operator topology. The spectral measure E satisfies similar properties as in the case of self-adjoint operators.

Spectral Theorem for Unitary Operators: The spectral Theorem for Unitary Operator is another specific case of the Spectral Theorem for Normal Operators. It applies to a unitary operator U on a Hilbert space H. It states there exists a unique spectral measure E on the Borel sets of the unit circle, denoted by Borel($\mathbb{T}$), such that U can be decomposed as:
$U = \int e^\wedge(i\theta) \, De(\theta)$
Where the integral is understood in the strong topology. The spectral measure E satisfies similar properties as in the case of self-adjoint and normal operators.

Relationship of the Spectrums
Spectral Theorem for Normal Operators: Let N be a normal operator on a complex inner product space H. The spectral theorem for normal operator on a complex inner product space H. The spectral theorem for normal operators states that N can be expressed as a decomposition of its eigenvalues and corresponding eigenvectors in the form:
$\mathbf{N} = \sum \lambda . P_i$
Where:
λ - represents the eigenvalues of N.
P_i - denotes the orthogonal projections onto eigenspace corresponding to eigenvalue.

Spectral Theorem for Self-Adjoint Operators: Let A be a self-adjoint operator on a complex inner product space H. The spectral theorem for self-adjoint operator on a complex inner product state that, A can be expressed as a decomposition of its eigenvalues and corresponding orthogonal eigenvectors in the form:
$A = \mathbf{N} = \sum \alpha . P_i$
Where:
α - represents the real eigenvalues.
P_i - Denotes the orthogonal projection onto the eigenspace corresponding to eigenvalues α_i

The key relationship between the two spectral theorems lies in the type of operators they apply to. Every self-adjoint operator is normal but not every normal operator is self-adjoint. In other words, the spectral theorem for self-adjoint operator is a special case of the spectral theorem for normal operators.

In summary, the spectral theorem for self-adjoint operators can be seen as specific instance of the more general spectral theorem for normal operators, the addition condition of self-ad jointness leads to real eigenvalues and orthogonal eigenvectors
Let's consider a bounded linear operator T on a complex Space X, and λ ∈ ℂ (the complex plane) is a scalar. The following symbols and notations can be used:

Continous spectrum (σ_c(T)): The continuous spectrum of the operator T is denoted by σ_c(T). It consists of all complex numbers λ for which (T - λI) does not have a bounded inverse, where I is the identity operator on X.
Symbolically, the continuous spectrum is defined as: σ_c(T) = {λ ∈ ℂ : (T - λI) does not have a bounded inverse.

Residual Spectrum (σ_r(T)): The residual spectrum of the operator T is denoted by σ_r(T). It consists of all complex numbers λ for which (T - λI) has a bounded inverse but is not a Fredholm operator (meaning its kernel and cokernel are infinite-dimensional).
Symbolically, the residual spectrum is defined as: : σ_r(T) = {λ ∈ ℂ : (T - λI) has a bounded inverse, but is not a Fredholm operator.

Point Spectrum (σ_p(T)): The point spectrum of the operator T is denoted by σ_p(T). It consists of all complex numbers λ for which (T - λI) does not have a bounded inverse, and its kernel(null space) is nontrivial.
Symbolically, the point spectrum is defined as: σ_p(T) = {λ ∈ ℂ : (T - λI) does not have a bounded inverse and Ker(T - λI) ≠ {0}}.

Continous Spectrum (σ_c(T)): Contains the λ for which (T - λI) does not have a bounded inverse.

Residual Spectrum (σ_r(T)): Contains the λ for which (T - λI) has a bounded inverse but is not a Fredholm operator.

Point Spectrum (σ_p(T)): Contains the λ for which (T - λI) does not have a bounded inverse, and its kernel is nontrivial.
In terms of size, the point spectrum (σ_p(T)) is the smallest among the continuous spectrum (σ_c(T)), residual spectrum σ_r(T)), and point spectrum (σ_p(T)).
Symbolically, we can express the relationships in terms of set inclusion as follows:
σ -p(T) ⊆σ_r(T)⊆ σ_c(T).

This means that every point in the point spectrum is also in the residual spectrum, and every point in the residual spectrum is also in the continuous spectrum. This makes sense because the point spectrum consists of the eigenvalues of the operator, which are isolated points in the spectrum. The residual spectrum contains values that are not eigenvalues but are still within the continuous spectrum. The continuous spectrum, being the largest, includes all the points that are not in the point or residual spectrum. In summary, the point spectrum is the smallest subset of the spectrum, followed by residual spectrum, and the continuous spectrum is the largest among the three.

CONCLUSION AND RECOMMENDATIONS

Conclusion: In conclusion, this study has explored the decomposition of the spectrum and the role of bounded linear mappings. This study has examined two key theorems: the Spectral Theorem for Self-Adjoint Operators and Spectral Theorem for Normal Operators. The Spectral Theorem for Self-Adjoint Operators establishes that every self-adjoint operator can be decomposed into a unique spectral measure defined on the Borel sets of its spectrum. This decomposition allows to understand in terms of its eigenvalues and eigenvectors. Furthermore, the Spectral theorem for Normal operators provide a more general framework for decomposition. It covers wider class of operators including self-adjoint operators as a special case, and allows for both real and complex eigenvalues. Throughout the investigation the study has recognised the significance of spectral measures and their role in decomposing operators. By understanding the spectral decomposition, we gain valuable information about operators' behaviour such as eigenvalues, eigenvectors and spectral properties. This knowledge can aid further applications, such as solving differential equations, studying quantum mechanics, or analysing signals in signal processing.

Recommendations: Investigate Advanced Theoretical Aspects: Future researchers should dive into advanced theoretical aspects of spectral decomposition and bounded linear mapping. This can include studying more specialised classes of operators, such as unbounded and infinite-dimensional operators, operator theory and operator algebras. By doing so, researchers can be in a position to gain deeper understanding of the underlying mathematical principles and develop new theoretical frameworks. Consider Numerical and Computational Methods: The future researchers should explore techniques that can handle large-scale problems, or address challenges specific to real-world applications. This intersection of theory and computation can lead to valuable insights and practical advancements. Bridge the gap between theory and practice: The future researchers should actively seek opportunities to bridge the gap between theoretical concepts and practical applications. This could involve collaborating with experts in specific domains and develop software tools that facilitate the implementation and utilisation of spectral decomposition and bounded linear mapping techniques in real world scenarios.

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