

CHARACTERISATION OF WEYL SPECTRUM IN HILBERT SPACES

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Abstract

In this paper, the study gives some continuity properties of the Weyl spectrum of a continuous linear operator on a Banach space and show that the Weyl's theorem holds for a spectral operator of finite type although the theorem fails for a spectral operator in general.

Key terms: Characterisation, continuity, dominance, Fredholm, Weyl theorem.

INTRODUCTION

In this study, some continuity properties of the Weyl spectrum of a continuous linear operator on a Banach space and show that the Weyl's theorem holds for a spectral operator of finite type although the theorem fails for a spectral operator in general.

Let $\mathfrak{N}(T)$ and $\mathfrak{R}(T)$ be respectively the null space and the range space of T . Let \mathfrak{F} be the class of Fredholm operators on X ($T \in \mathfrak{F}$ if and only if $\mathfrak{R}(T)$ is closed and dimension $\mathfrak{N}(T)$ and co-dimension $\mathfrak{N}(T)$ are both finite) and let \mathfrak{F}_0 be the class of Fredholm operators of index 0 (i.e., dimension $\mathfrak{N}(T) = \text{co-dimension } \mathfrak{R}(T)$).

This study will also focus on dominance and continuity of Weyl spectrum to expose on the significance of upper and lower semi-continuity mapping. It will be very interesting to study contribution of limit superior and inferior affects the neighborhoods of λ .

Preliminaries

Let H be a complex Hilbert space. By an operator on H , we shall mean a bounded linear transformation from H to H . Let, and respectively denote the spectrum, the approximate point spectrum, the point spectrum, the set eigenvalues with finite multiplicity and the Weyl spectrum of an operator. If for an operator, then we say that the Weyl's theorem holds for. The spectral radius and the numerical radius of will be denoted by and respectively. If, then is exists some such that for each positive integer.

Also, in this paper X will denote a complex Banach space and $\mathcal{L}(X)$ the space of continuous linear operators on X considered with the norm topology multiplicity. Let $\mathfrak{N}(T)$ and $\mathfrak{K}(T)$ be respectively the null space and the range space of T . Let \mathfrak{F} be the class of Fredholm operators on X ($T \in \mathfrak{F}$ if and only if $\mathfrak{K}(T)$ is closed and dimension $\mathfrak{N}(T)$ and co-dimension $\mathfrak{N}(T)$ are both finite) and let \mathfrak{F}_0 be the class of Fredholm operators of index (i.e., dimension $\mathfrak{N}(T) = \text{co-dimension } \mathfrak{K}(T)$). Let $\mathfrak{C}(X)$ be the ideal of compact operators on X and let \hat{T} be the image of T under the canonical mapping of $\mathcal{L}(X)$ into the quotient algebra $\mathcal{L}(X)/\mathfrak{C}(X)$. Finally, let \mathbb{C} be the set of complex numbers.

In the study, the following **Theorems** will help expose the characterisation of the Weyl spectrum:

1. The mapping $T \rightarrow \omega(T)$ is upper semi-continuous at T . **(Theorem 1)**.
2. If $T_n \rightarrow T$. Then if $\lim \sigma(\hat{T}_n) = \sigma(\hat{T})$ then $\lim \sigma(T_n) = \omega(T)$. **(Theorem 2)**
3. We Let S be a scalar type operator on X . Then Weyl's theorem holds for S . **(Theorem 3)**
4. Let T be a spectral operator of finite type. Then Weyl's theorem holds for T . **(Theorem 4)**

Lemmas will also help us in this study;

1. If T is a quasi-nilpotent operator and hence a spectral operator then $0 \in \pi_{00}(T)$ and also $0 \in \omega(T)$. Thus T does not satisfy the weyl's theorem **(Lemma 1)**.
2. The operator T has a closed range if and only if
 - (i) the point $\lambda = 0$ is either in $\mathfrak{p}(T)$ or is an isolated point of $\sigma(T)$, and
 - (ii) the operator $TE(\{0\})$ has a closed range **(Lemma 2)**.

3. Let T be a spectral operator of finite type so that for some non-negative integer m , $N^m = 0$. Then $\pi_{00}(S) = \pi_{00}(T)$. (**Lemma 3**).

Note, we are now ready to do the study using the theorems and lemmas stated above.

Definition 1

The Weyl spectrum $\omega(T)$ of $T \in \mathcal{E}(X)$ is defined by

$$\omega(T) = \{\lambda \in \mathcal{C}: \lambda I - T \notin \mathfrak{F}_0\}$$

It is well known that

- (i) $T \in \mathfrak{F}$ if and only if $0 \in \rho(\hat{T})$, and
- (ii) $\sigma(\hat{T}) \subset \omega(T) \subset \sigma(T)$.

In particular if X is infinite dimensional then $\omega(T)$ is non-empty compact subset of \mathcal{C}

Dominance and Continuity of $\omega(T)$

This section defines upper and lower semi-continuity of mapping $T \rightarrow \omega(T)$

And show that this mapping is upper semi-continuous while it may not be lower semi-continuous.

Definition 2

Let (G_n) be a sequence of compact subsets of \mathcal{C} . The *limit inferior*, $\liminf G_n$ is the set of all λ in \mathcal{C} such that every neighbourhood of λ has a non-empty intersection with all but finitely many G_n . The *limit superior*, \limsup is the set of all λ in \mathcal{C} such that every neighbourhood of λ intersects infinitely many G_n . If $\liminf G_n = \limsup G_n$ then $\lim G_n$ is said to exist and is equal to this common limit.

A mapping τ defined on $\mathcal{E}(X)$ whose values are compact subsets of \mathcal{C} is said to be *upper semi-continuous* at T when if $T_n \rightarrow T$ then $\limsup \tau(T_n) \subset \tau(T)$. τ is *lower semi-continuous* at T if $\tau(T) \subset \liminf \tau(T_n)$. If τ is both upper and lower semi-continuous at T then it is said to be *continuous* at T and in this case $\lim \tau(T_n) = \tau(T)$.

Theorem 1

The mapping $T \rightarrow \omega(T)$ is upper semi-continuous at T .

Proof

Let $\lambda \notin \omega(T)$ so that $\lambda I - T$ is a Fredholm operator of index 0, there exists an $\eta > 0$ such that if $S \in \mathcal{L}(X)$ and $\|\lambda I - T - S\| < \eta$ then $S \in \mathfrak{F}_0$.

There exists an integer N such that $\|\lambda I - T - (\lambda I - T_n)\| < \eta/2$ for $n \leq N$.

Let V be an open $(\eta/2)$ neighbourhood of λ . We have, for $\mu \in V$ and $n \geq N$
 $\|\lambda I - T - (\mu I - T_n)\| < \eta$

So that $(\mu I - T_n) \in \mathfrak{F}_0$. This implies that $\lambda \notin \limsup \omega(T_n)$. Thus, $\limsup \omega(T_n) \subset \omega(T)$ and the theorem is proved.

To show that the mapping $T \rightarrow \sigma(T)$ is in general not lower semi-continuous may be used to show that the mapping $T \rightarrow \omega(T)$ need to be lower semi-continuous.

Theorem 2

Let $T_n \rightarrow T$. Then if $\lim \sigma(\hat{T}_n) = \sigma(\hat{T})$ then $\lim \sigma(T_n) = \omega(T)$.

Proof

In the presence of Theorem 1 it is enough to show that $\omega(T) \subset \liminf \omega(T_n)$.

Suppose $\lambda \notin \liminf \omega(T_n)$ so that there is a neighbourhood V of λ that does not intersect infinitely many $\omega(T_n)$. Since does not intersect infinitely many $\sigma(\hat{T}_n)$, i. e., $\lambda \notin \lim \sigma(\hat{T}_n) = \sigma(\hat{T})$. This shows that $(\lambda I - T) \in \mathfrak{F}$. Now it is easy to see that $\text{index}(\lambda I - T) = 0$ so that $\lambda \in \omega(T)$

Corollary 1

Let $T_n \rightarrow T$. Then $\lim \omega(T_n) = \omega(T)$ in each one of the following cases.

- (i) $T_n T = T T_n$ for all n .
- (ii) $\sigma(T)$ is totally disconnected.
- (iii) X is a Hilbert space and T, T_n are normal operators.

Proof

Each one of the above conditions implies $\lim \sigma(\hat{T}_n) = \sigma(\hat{T})$

Weyl's Theorem

Let $T \in \mathcal{L}(X)$. If

$$(*) \quad \omega(T) = \sigma(T) \sim \pi_{00}(T)$$

Then we say that Weyl's theorem holds for T . If X is finite dimensional then, of course, Weyl's theorem holds for each $T \in \mathcal{L}(X)$. There are several classes of operators including normal and hyponormal operators on a Hilbert space (see e.g., [2] and [3] for which Weyl's theorem holds. This section shows that if T is a spectral operator, in the sense of Dunford, of finite type.

LEMMA 1

If T is a quasi-nilpotent operator and hence a spectral operator then $0 \in \pi_{00}(T)$ and also $0 \in \omega(T)$. Thus T does not satisfy the Weyl's theorem.

Proof

Let $T \in \mathcal{L}(X)$. If

$$\omega(T) = \sigma(T) \sim \pi_{00}(T) \quad \text{_____ (a)} \quad \text{and so}$$

$$\sigma(T) \sim \omega(T) = \pi_{00}(T) \quad \text{which implies} \quad \pi_{00}(T) \cup \omega(T) = \sigma(T)$$

And if $0 \in \pi_{00}(T)$ and also $0 \in \omega(T)$ then it's clear that it does not satisfy (a) above. **Q.E.D.**

Remark

There are several classes of operators including normal and hyponormal operators on a Hilbert spaces for which Weyl's theorem holds.

Lemma 2

The operator T has a closed range if and only if

- (iii) the point $\lambda = 0$ is either in $p(T)$ or is an isolated point of $\sigma(T)$, and
- (iv) the operator $TE(\{0\})$ has a closed range.

Theorem 3

Let S be a scalar type operator on X . Then Weyl's theorem holds for S .

Proof

We have to show that $\lambda \in \pi_{00}(S)$ if and only if $\lambda \in \sigma(S) \sim \omega(S)$. without loss of generality we may assume that $\lambda = 0$.

Let $0 \in \pi_{00}(S)$ so that $\mathfrak{N}(S)$ is finite dimensional and by Lemma 2, $\mathfrak{N}(S)$ is closed. Lemma 1 shows that $Sx = 0$ if and only if $Sx = 0$. Hence, $\mathfrak{N}(S) \cap \mathfrak{R}(S) = \{0\}$.

Also, from the relation $\sigma(S|_{E(\Delta)X \subset \Delta})$ for a Borel subset Δ of \mathbb{C} it is easy to see that $\mathfrak{N}(S) \oplus \mathfrak{R}(S) = X$.

Thus dimension $\mathfrak{N}(S) = \text{codimension } \mathfrak{R}(S)$ so that $S \in \mathfrak{F}_0$ i.e., $0 \notin \omega(S)$.

Conversely suppose $0 \in \sigma(S) \sim \omega(S)$. Since $\mathfrak{N}(S)$ is closed, 0 is an isolated point of $\sigma(S)$. Also $\mathfrak{N}(S)$ is finite dimensional and non-zero so that $0 \in \pi_{00}(S)$.

Corollary 2

For operator S on X , $\lambda \in \pi_{00}(S)$ if and only if $\lambda \in \sigma(S) \sim \omega(S)$.

Proof

Without loss of generality we assume that $\lambda = 0$ and then let $0 \in \pi_{00}(S)$

So that $\mathfrak{N}(S)$ is closed then;

$\mathfrak{N}(S) \oplus \mathfrak{R}(S) = X$ so that for a Borel subset Δ of \mathbb{C} it is easy to see that

$\mathfrak{N}(S) \oplus \mathfrak{R}(S) = X$ and hence it is easy to see that $S \in \mathfrak{F}_0$ which implies that $0 \notin \omega(S)$.

Conversely; let $0 \in \sigma(S) \sim \omega(S)$ Since $\mathfrak{N}(S)$ is closed, 0 is an isolated point of $\sigma(S)$ then $0 \in \pi_{00}(S)$. Q.E.D.

Remark; If the dimension of the null space is equal to co-dimension of the range space, then we state the following corollary.

Corollary 3

The nullity and defect conditions exist for the spectral operator T and are equal. i.e. nullity = defect for the Weyl conditions.

Lemma 3

Let T be a spectral operator of finite type so that for some non-negative integer m , $N^m = 0$. Then $\pi_{00}(S) = \pi_{00}(T)$.

Proof

We need only to show that $0 \in \pi_{00}(S)$ if $0 \in \pi_{00}(T)$. Let $0 \in \pi_{00}(S)$. It is immediate that if $Sx = 0$ then $T^m x = 0$. Thus, 0 is an eigenvalue of T . From the relation $\mathfrak{R}(T) \subset \mathfrak{R}(S)$ it follows that $0 \in \pi_{00}(T)$.

Conversely let $0 \in \pi_{00}(T)$ so that 0 is also an eigenvalue of S . Since $\mathfrak{R}(T)$ is a finite-dimensional subspace of $\mathfrak{R}(S)$ we may write

$$\mathfrak{R}(S) = \mathfrak{R}(T) \oplus Y.$$

If $y \in Y$ then $Sy = 0$ so that $T^m y = 0$ i.e., $T^{m-1} y \in \mathfrak{R}(T)$. This implies that Y and hence $\mathfrak{R}(S)$ is finite dimensional showing thereby that $0 \in \pi_{00}(S)$.

Theorem 4

Let T be a spectral operator of finite type. Then Weyl's theorem holds for T .

Proof

We have $\omega(S) = \sigma(S) \sim \pi_{00}(S) = \sigma(T) \sim \pi_{00}(T)$.

Hence, the theorem follows if we show that $\omega(S) = \omega(T)$. It is enough to show that $0 \in \omega(S)$ if and only $0 \in \omega(T)$.

Let $0 \notin \omega(S)$ so that $S \in \mathfrak{F}_0$. Since $\mathfrak{R}(S)$ is closed, either $0 \in p(S) = p(T)$, or 0 is an isolated point of $\sigma(S) = \sigma(T)$ and $\mathfrak{R}(S) = E(\{0\})X$ is finite dimensional. Therefore $TE(\{0\})X$ is finite dimensional and hence a closed subspace of X . By Lemma 2, $\mathfrak{R}(T)$ is closed.

Let

$$(1) X = \mathfrak{R}(S) \oplus Y \text{ where } Y = \mathfrak{R}(S) = E(C \sim \{0\})X.$$

Also, let

$$(2) \mathfrak{R}(S) = \mathfrak{R}(T) \oplus \text{span} \{x_1, x_2, \dots, x_r\},$$

where x_1, \dots, x_r are linearly independent. It is easy to verify that Tx_1, \dots, Tx_r are linearly independent.

We assert that

$$(3) \mathfrak{R}(T) = Y \oplus \text{span} \{Tx_1, \dots, Tx_r\}.$$

Since $0 \notin \sigma(T|Y)$, $TY = Y$. If possible let $Tx_i = y \in Y$ for some $i (1 \leq i \leq r)$. Since S is injective on Y we have

$$0 \neq Sy = STx_i = 0$$

which is a contradiction. In fact no non-zero linear combination of Tx_i can belong to Y .

This proves our assertion. Relations (1), (2) and (3) together with the fact that $S \in \mathfrak{F}_0$ show that $T \in \mathfrak{F}_0$ i.e., $0 \in \omega(T)$.

The converse assertion viz., if $0 \in \omega(T)$ then $0 \in \omega(S)$ follows in exactly the same fashion.

RESULTS AND DISCUSSION

From the study, the following results are very important;

Theorem

The mapping $T \rightarrow \omega(T)$ is upper semi-continuous at T .

Proof

Let $\lambda \notin \omega(T)$ so that $\lambda I - T$ is a Fredholm operator of index 0, there exists an $\eta > 0$ such that if $S \in \mathcal{L}(X)$ and $\|\lambda I - T - S\| < \eta$ then $S \in \mathfrak{F}_0$.

There exists an integer N such that $\|\lambda I - T - (\lambda I - T_n)\| < \eta/2$ for $n \leq N$.

Let V be an open $(\eta/2)$ neighbourhood of λ . We have, for $\mu \in V$ and $n \geq N$
 $\|\lambda I - T - (\mu I - T_n)\| < \eta$

so that $(\mu I - T_n) \in \mathfrak{F}_0$. This implies that $\lambda \notin \limsup \omega(T_n)$. Thus $\limsup \omega(T_n) \subset \omega(T)$ and the theorem is proved.

To show that the mapping $T \rightarrow \sigma(T)$ is in general not lower semi-continuous may be used to show that the mapping $T \rightarrow \omega(T)$ need to be lower semi-continuous. **Q.E.D.**

Note; Them we come up with following corollary

Corollary

Mapping $T \rightarrow \sigma(T)$ is in general not lower semi-continuous

Theorem

Let $T_n \rightarrow T$. Then if $\lim \sigma(\hat{T}_n) = \sigma(\hat{T})$ then $\lim \sigma(T_n) = \omega(T)$.

Proof. In the presence of Theorem 1 it is enough to show that $\omega(T) \subset \liminf \omega(T_n)$.

Suppose $\lambda \notin \liminf \omega(T_n)$ so that there is a neighbourhood V of λ that does not intersect infinitely many $\omega(T_n)$. Since does not intersect infinitely many $\sigma(\hat{T}_n)$, i. e., $\lambda \notin \lim \sigma(\hat{T}_n) = \sigma(\hat{T})$. This shows that $(\lambda I - T) \in \mathfrak{F}$. Now it is easy to see that index $(\lambda I - T) = 0$ so that $\lambda \notin \omega(T)$

Corollary

Let $T_n \rightarrow T$. Then $\lim \omega(T_n) = \omega(T)$ in each one of the following cases.

- i) $T_n T = T T_n$ for all n .
- ii) $\sigma(T)$ is totally disconnected.
- iii) X is a Hilbert space and T, T_n are normal operators.

Proof

Each one of the above conditions implies $\lim \sigma(\hat{T}_n) = \sigma(\hat{T})$

We conclude this paper with the following corollary.

Corollary

The Weyl's theorem holds for T a finite spectral operator such that $\omega(S) = \omega(T)$.

Also we note that The nullity and defect conditions exists for the spectral operator T and are equal. ie nullity=defect for the weyl conditions.

Its also clear to note that if we: *Let T be a spectral operator of finite type. Then Weyl's theorem holds for T .*

Proof

We have $\omega(S) = \sigma(S) \sim \pi_{00}(S) = \sigma(T) \sim \pi_{00}(T)$.

Hence, the theorem follows if we show that $\omega(S) = \omega(T)$. It is enough to show that $0 \in \omega(S)$ if and only $0 \in \omega(T)$.

Let $0 \notin \omega(S)$ so that $S \in \mathfrak{F}_0$. Since $\mathfrak{R}(S)$ is closed, either $0 \in p(S) = p(T)$, or 0 is an isolated point of $\sigma(S) = \sigma(T)$ and $\mathfrak{R}(S) = E(\{0\})X$ is finite dimensional **Q.E.D.**

RECOMMENDATION

We need Study further the connection of weyl spectrum in connection to both dominance and characterisation.

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