



DECOMPOSITION AND PROPERTIES OF A SPECTRUM

Author

Peter Githara Rugiri 

Email: githarapeterrugiri@yahoo.com

Kabarak University, Kenya.

Cite this article in APA

Rugiri, P. G. (2023). Decomposition and Properties of a Spectrum. *Journal of physical and applied sciences*, 2(1), 55-64. <https://doi.org/10.51317/jpas.v2i1.394>



A publication of Editon Consortium Publishing (online)

Article history

Received: 25.06.2023

Accepted: 18.07.2023

Published: 30.08.2023

Scan this QR to read the paper online



Copyright: ©2023 by the author(s).

This article is an open access article distributed under the license of the Creative Commons Attribution (CC BY) and their terms and conditions.



Abstract

The study sought to show the relationships of the decomposed subsets of the spectrum by use of inclusions. In operator Theory, we study of linear operator on function spaces acting on vector spaces of functions. Since an element of $\sigma(A)$ (ie spectrum of A), is a spectrum value of A, we need to investigate more about these elements of A, by decomposing the spectrum to various subsets. From the decomposed subsets of the spectrum, we study the properties of spectrum to give an insight of deeper understanding of the spectrum. The deductions from the proofs of Theorems, propositions, Lemmas and corollaries will help to expose the properties being investigated.

Key terms: Decomposition, Linear Operator, Properties, Spectrum.

INTRODUCTION

In operator Theory, it is known that if A and B are operators with at least one of them invertible then AB and BA are similar operators. We introduce scalars $\lambda \in \mathbb{C}$, by stating the following definition. $\sigma(A) = \{\lambda \in \mathbb{C}, (A - \lambda I) \text{ is not invertible in } B(H)\}$

An element of $\sigma(A)$ is called a spectral value of A and hence finding the spectrum of a bounded operator involves invertibility of certain elements of $B(H)$ (where $B(H)$ is a set of bounded operators in a complex Hilbert space).

Remark 1

Since an element of $\sigma(A)$ (ie spectrum of A), is a spectrum value of A, we need to investigate more about these elements of A, and so we define the eigen values.

Definition 1

Let A be a transformation from V to V where V is a finite – dimensional vector space, let $v \in V$ such that $Av = \lambda v$ where λ is a scalar for all $v \in V$, λ is called an eigenvalue corresponding to the linear transformation A. If $v \in V$ such that $v \neq 0$ and $Av = \lambda v$ is referred to as eigenvector corresponding to eigen value.

Definition 2

The set of all eigenvalues of $A \in B(H)$ is called the eigen spectrum.

Theorem 1

Every non-zero spectrum value of A is an eigenvalue of A (Limaye, 1981).

Proof

Let $0 \neq \lambda \in \mathbb{C}$ and $\lambda \in \sigma(A)$

Since $A - \lambda I$ is not invertible, it is either not one to one or onto. Then $A - \lambda I$ is not one to one hence $\lambda \in$ eigen spectrum. **Q.E.D.**

The name eigen spectrum arises from the following physical considerations. If a physical quantity (like position, momentum or energy) represented by an operator A is measured in an experiment, then the result of the measurement is one of the eigenvalues of A. In an atomic quantum mechanical system if A is the energy operator of an atom, then the differences of the various eigenvalues of A are the amounts of energy emitted by the atom as it undergoes a transition. The amounts are seen in form of electromagnetic waves, which constitute the optical spectrum of that atom.

Theorem 2

Let H be a Hilbert space over finite dimension n. Then the spectrum of every operator on H consists of n eigenvalues. If H is a Hilbert space over R, then the spectrum of every self-adjoint operator on H consists of n real eigenvalues (Limaye, 1981).

Definition 2

Let us consider an equation of the form $f(x) = 0$ where $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$, $a_0 \neq 0$. In some algebraic extension field K of k $f(x)$ can be factored.

$$f(x) = a_0(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_n).$$

$\lambda_1, \dots, \lambda_n$ are called the roots of equation of degree n has exactly n roots. If λ appears ρ times in $\lambda_1, \dots, \lambda_n$, we say that λ is a ρ -tuple root, and ρ is called the multiplicity of the root λ . When $\rho=1$, λ is called a simple root when $\rho \geq 2$, λ is called a multiple root.

Definition 3

The dimension of the Eigen spectrum corresponding to the eigenvalue λ_0 is called the multiplicity of the eigenvalue λ_0 . Eigenvalues which occurs exactly n – times is said to be of multiplicity n .

Example 1

If the characteristic equation of a given 6×6 matrix is $(3-\lambda)^2(1+\lambda)^3(2-\lambda) = 0$

Then 3 is an eigenvalue with multiplicity 2, and -1 is an eigenvalue with multiplicity 3 (and 2 is a 'normal' eigenvalue).

Theorem 3

Bollobás (1990) suggests that if the operator A has distinct eigenvalues then there exists a complete set of linearly independent eigenvectors.

Remark 2

We may not be particularly interested in the value of eigenvalues but rather in its relative position in the spectrum and this lead us to study the spectrum and the resolvent set.

Definitions 4

We say that an operator A is bounded below on H if there is a constant $\alpha > 0$ such that

$$\alpha \|x\| \leq \|A(x)\| \text{ for all } x \in H$$

Lemma 1

$A \in B(H)$ is invertible in $B(H)$ if A is bounded below and the range of A is dense in H Bollobás (1990).

Proof

Let A be invertible for $x_0 \in H$, if $A(x) = y$,

$$\text{Then } \|x\| = \|A^{-1}(y)\| \leq \|A^{-1}\| \|y\| = \|A^{-1}\| \|A(x)\|.$$

Hence A is bounded below. Also, the range of A is dense in H ,

Conversely, assume that;

$$\alpha \|x\| \leq \|A(x)\| \text{ for all } x \in H \text{ and some constant;}$$

$\alpha > 0$, and that the range A is dense in H . To show that A is onto, it is enough to prove that the range of A is closed in H .

Let $A(x_n) \rightarrow y$ in H

Then with $y_n = A(x_n)$ we have for all n, m ,

$$\|x_n - x_m\| \leq (1/\alpha) \|A(x_n - x_m)\| = 1/\alpha \|y_n - y_m\|.$$

Hence, (x_n) is a Cauchy sequence in H , since H is complete, let $x_n \rightarrow x \in H$. Then, by continuity of A , $A(x_n) \rightarrow A(x)$ so that $y = A(x)$ is in the range of A . Thus, A is onto. Since A is bounded below, it is clearly one to one. Let A^{-1} be the set theoretic inverse of A on H . Then A^{-1} is seen to be automatically linear. Also, for any $y \in H$,

If $y = A(x)$ then;

$$\|A^{-1}(y)\| = \|x\| \leq \frac{1}{\alpha} \|A(x)\| = \frac{1}{\alpha} \|y\|$$

Thus, A^{-1} is a bounded operator on H ; ie, A is invertible in $B(H)$.

Q.E.D.

Remark 3

It is clear that for $A \in B(H)$ which is bounded below and has a dense range is invertible, and so we can now state the following definition

Definition 5

If λ_0 is such that the range $R \sigma(A)$ is dense and $\sigma(A)$ has a continuous inverse, $(\lambda_0 I - A)^{-1}$, we say that λ_0 is in the resolvent set denoted $\rho(A)$ of A , and denote this inverse $(\lambda_0 I - A)^{-1}$ and call it the resolvent at λ_0 .

The complex number λ is called a regular point of the operator A if the operator $(A - \lambda I)$ has an inverse $(A - \lambda I)^{-1}$. In the opposite case, λ is called a point of the spectrum of the operator A .

If λ is a regular point of A , then the bounded linear operator $(A - \lambda I)^{-1}$ is called a resolvent.

The regular points form an open set in the complex plane; the spectrum is closed.

All points lying outside the circle of radius $\|A\|$ with center at the origin are regular. All points of the spectrum are in the disk;

$$|\lambda| \leq \|A\|.$$

The radius of the smallest disk with center at the origin containing the spectrum of the operator A is called spectral radius r_A of the operator A .

The series for the resolvent will converge if $r_A < |\lambda|$ and diverge if $r_A > |\lambda|$ in particular, the series.

$$(I - A)^{-1} = -R_1 = I + A + A^2 + \dots + A^n + \dots$$

Converges if $r_A < 1$ and diverges if $r_A > 1$ operator is a non-empty set. If

The spectrum of an arbitrary bounded operator is a non-empty set then

$$r_A = \lim_{n \rightarrow \infty} \sqrt[n]{\|A^n\|} = 0; \text{ then}$$

the spectrum consists of one point, $\lambda = 0$.

Definition 6

An operator-valued function of λ is called analytic at λ_0 if it can be expanded in a neighbourhood of λ_0 in a series of positive integral powers of $(\lambda - \lambda_0)$ which converges with respect to the operator norm.

The resolvent is an operator-valued function of λ which is analytic in the region consisting of regular points of the operator A .

Example 2

For $x \in E$, the function $\rho_\lambda(x)$, whose domain consists of regular points of A , is analytic function with values in E .

Let λ_0 be a pole of analytic function ρ_λ . Then any element $\rho_\lambda(x)$ has an expansion

$$= f_0 + f_1(\lambda - \lambda_0) + \dots + f_n(\lambda - \lambda_0)^n + \dots$$

The element $\rho_0 = \lambda \rho_0(x)$ satisfies the equation $A\rho_0 = \lambda_0\rho_0$

And is called an eigenvector of the operator A corresponding to the eigenvalue λ_0 .

Remark 4

We can now state the following definition on resolvent.

Definition 7

A point $\lambda \in \mathbb{C}$ is said to belong to the resolvent set of A , if $R(\lambda I - A) = x$ and $(\lambda I - A)^{-1}$ exists and is bounded where I is the identity operator. The inverse operator $(\lambda I - A)^{-1}$ is called the resolvent set.

Definition 8

The complement of $\rho(A)$ in \mathbb{C} ie the set of all $\lambda \in \mathbb{C}$, such that $\lambda \in \rho(A)$ is called the spectrum of A denoted by $\sigma(A)$.

ie $\sigma(A) = (\rho(A))^c$

Theorem 4

A complex number $\lambda \in \sigma_p$ (where $\sigma_p(A)$ is point spectrum equivalent to eigen spectrum if the equation $Ax = \lambda x - I$ _____ (a) ; has a non-zero solution of x (Douglas, 1969).

Proof

Let $\lambda \in \sigma_p(A)$ then its very clear

That $(\lambda I - A)^{-1}$ does not exist and

So $(\lambda I - A)$ is not one to one so

$(\lambda I - A)x = 0$ for some $x \neq 0$

$\lambda I - Ax = 0$ for some $x \neq 0$. Therefore $Ax = \lambda x$ has a non-zero solution in x ,

Conversely, if $Ax = \lambda x$ for some $x \neq 0$

But $(\lambda I - A)x = 0$.

Therefore $\lambda I - A$ is not one to one ie $(\lambda I - A)^{-1}$ does not exist

ie $\lambda \in \sigma_p(A)$

Q.E.D.

Remark 5

The above theorem exhibits a very important property of the spectrum, which can be shown by stating the following theorem.

Theorem 5

Douglas (1969) suggests that the spectrum of an operator $A \in B(H)$ is not empty.

Proof

Suppose that $\sigma(A) = \emptyset$, let $R(\lambda)$ denote the element of resolvent at λ .

Since $R(\lambda)$ is an analytic function on the resolvent which now the entire plane, $R(\lambda)$ is an entire function
 Furthermore since $\|R(\lambda)\| = 0$
 As $|\lambda| \rightarrow \infty$
 Hence before Louisville's theorem $R(\lambda)$ is constant and $R(\lambda) = 0$ for every λ .
 But this is clearly impossible. **Q.E.D.**

Remark 6

We are now ready to decompose the spectrum into various subsets.

Definition 8

Point spectrum is defined and denoted as follows;

$$\sigma_p(A) = \{\lambda \in \mathbb{C}, : \ker (\lambda I - A) = (0)\}.$$

The elements of $\sigma_p(A)$ are eigenvalues

Clearly;

$$\sigma_p(A) \subset \sigma(A) \text{ _____ (1)}$$

Hence;

$$\sigma(A) = \sigma_p(A) \text{ for every operator on a finite-dimensional space _____ (1*)}$$

Definition 1.4.25

Let approximate point spectrum be denoted by $\sigma_{ap}(A)$. Hence;

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} \text{ the sequence } (x_n) \text{ such } (\lambda I - A) x_n \rightarrow 0\}$$

Sometimes (x_n) is called an approximate eigenvector with eigenvalues λ

$$\text{Clearly } \sigma_p(A) \subset \sigma_{ap}(A) \text{ _____ (2)}$$

Remark 7

We now state a very important theorem as a result of (1) and (2) above

Theorem 6

Let $A \in B(H)$

$$\sigma_p(A) \subset \sigma_{ap}(A) \subset \sigma(A) \text{ (Douglas, 1969).}$$

Proof

If $\lambda \in \sigma(A)$, then let x be an eigenvector corresponding to eigenvalue λ such that;

$$\|x\| = 1, \text{ and let}$$

$x_n = x$ for $n = 1, 2, \dots$, then

$$0 = \| A(x_n) - \lambda x_n \| \rightarrow 0,$$

So that

$$\lambda \in \sigma_{ap}(A)$$

On the other hand, if $\lambda \in \sigma(A)$, then for every $x \in X$ with $\|x\| = 1$,

$$1 = \|x\| = \| (A-\lambda I)^{-1} 0(A-\lambda I)x \| \leq \| (A-\lambda I)^{-1} \| \| (A(x)-\lambda(x)) \|$$

So that $\lambda \notin \sigma(A)$

and thus;

$$\sigma_p(A) \subset \sigma_{ap}(A) \subset \sigma(A) \text{ follows.}$$

Q.E.D.

Definition 9

Continuous spectrum denoted by $\sigma_c(A)$ is the totality of a complex numbers λ for which $\sigma(A)$ has discontinuous inverse with domain being dense.

Thus;

$$\sigma_c(A) = \sigma(T) \setminus \sigma_{com}(A) \cup \sigma_p(T) \rightarrow \sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$$

Where $\sigma_r(A)$ is the residue spectrum defined as follows;

Definition 10

The residue spectrum is defined and denoted by

$$\sigma_r(A) = \sigma_{com}(A) \setminus \sigma_p(A)$$

$$\text{Note } \sigma(A) = \sigma_{ap}(A) \cup \sigma_{com}(A)$$

Where $\sigma_{com}(A)$ is the compression spectrum which is the set of all λ such that λ belongs to $\sigma(A) - \sigma_{ap}(A)$

Theorem 8

According to Douglas (1969), if x is a finite dimensional normed linear space and $A: x \rightarrow x$ is linear operator then

$$\sigma_r(A) = \phi, \sigma_{com}(A) = \phi \text{ Thus } \sigma(A) = \sigma_p(A).$$

Proof

We first note that every; linear transformation of a finite dimensional normed linear space is bounded and thus for all $\lambda \in \mathbb{C}$, $(\lambda I - A)^{-1}$ is bounded if it exists on;

$$R(\lambda I - A) \text{ is bounded.} \quad (*)$$

Thus $\sigma_{com}(A) = \phi$ for its existence asserts that $(\lambda I - A)^{-1}$ is unbounded which contradicts the (*) above

Now for $\lambda \in \sigma_r(A)$, $(\lambda I - A)^{-1}$ exists and is bounded for x has finite dimensions. Therefore $(\lambda I - A)$ is one to one of x

Let $\{x_1, x_2, \dots, x_n\}$ be a basis of x .

Then since $(\lambda I - A)$ is one to one it follows that the set $\{(\lambda I - A)x_1, (\lambda I - A)x_2, \dots, (\lambda I - A)x_n\}$ which spans $R(\lambda I - A)$ is linearly independent. Thus $R(\lambda I - A) = x$

Therefore if $\lambda \in \sigma_r(A)$ then

$$R(\lambda I - A) = x. \text{ But this contradicts the definition of } \sigma_r(A). \text{ Thus there is no such } \lambda.$$

Hence $\sigma_r(A) = \phi$

Theorem 9

Let $A \in B(H)$, then $\lambda \in \sigma_{ap}(A)$ if and only if $(\lambda I - A)$ does not have a bounded inverse on H (Bollobás (1990)).

Proof

Suppose $\lambda \in \sigma_{ap}(A)$, then for each $n \in \mathbb{J}^+ \exists x_n \in D(A)$ where $D(A)$ is the domain of A with $\|x_n\| = 1$ such that;

$$\|(\lambda I - A)x_n\| < 1/n$$

Thus it is not possible to find $k > 0$ such that $\|(\lambda I - A)x\| \geq k\|x\|$

$\forall x \in D(A)$

ie $(\lambda I - A)$ is not bounded from below

Thus $\lambda I - A$ does not have a bounded inverse on H .

Conversely, let $\lambda I - A$ to have no bounded inverse. In this case $k > 0$ satisfying.

$$\|(\lambda I - A)x\| \geq k\|x\| \quad \forall x \in D(A)$$

does not exist. This means that for any $\epsilon > 0$ an $x \in D(A)$ with $\|x\| = 1$ can be found such that $\|(\lambda I - A)x\| < \epsilon$

This implies that $\lambda \in \sigma_{ap}(A)$

Q.E.D.

Remark 7

From the earlier result already shown by definition 2.12, we conclude that $\sigma_{ap}(A) \subset \sigma(A)$

This can be stated as a corollary.

Corollary 1

$$\sigma_{ap}(A) \subset \sigma(A)$$

Proof

Let $\lambda \in \sigma(A)$

Let denote are solvent set by $\rho(A)$

Then $\lambda \notin \sigma(A) \rightarrow \lambda \in \rho(A)$

$\rightarrow \lambda I - A$ has a bounded inverse

$\rightarrow \lambda \notin \sigma_{ap}(A)$ (as above).

ie $\lambda \notin \sigma(A) \rightarrow \lambda \notin \sigma_{ap}(A)$

ie $\lambda \in \sigma_{ap}(A) \rightarrow \lambda \in \sigma(A)$

Q.E.D.

Remark 8

In the introduction we stated that a numerical range as a set whose closure is known to contain spectrum, we now study the properties of spectrum and numerical range.

Definition 11

The numerical range of A is defined and denoted as follows

$W(A) = \{\lambda \in \mathbb{C} : \lambda = \langle Ax, x \rangle \text{ for some } x \in H \text{ with } \|x\| = 1\}$

One of the most interesting and surprising facts about the numerical range of any bounded operator on a complex Hilbert space is that it is a convex set. The line segment joining any two points in it, is itself contained in it.

Theorem 11

Let $A \in B(H)$ where it is finite dimensional;

Then;

$\sigma(A) \subseteq W(A)$ (Douglas, 1969).

Remark 9

From the theorem 11 the topological property of the spectrum and numerical range can be exhibited by the following theorem.

Theorem 12

The eigenvalues of every operator A belong to $W(A)$ (Halmos, 1967).

Proof

If $Ax = \lambda x$ with $\|x\| = 1$, then $\langle Ax, x \rangle = \lambda$

If A is normal then;

$\|A\| = \sup \{|\lambda| : \lambda \in W(A)\}$

So that there always exists a λ in $W(A)$ such that $|\lambda| = \|A\|$

It follows that if a normal operator has sufficiency many eigenvalues to approximate its norm, but does not have one whose module is as large as the norm, then its numerical range will not be closed.

RESULTS AND CONCLUSION

From the study, the following results are true;

$\sigma(A) = \sigma_p(A)$ for every operator on a finite-dimensional space.

One of the most interesting and surprising facts about the numerical range of any bounded operator on a complex Hilbert space is that it is a convex set. The line segment joining any two points in it, is itself contained in it, and thus;

$\sigma_p(A) \subset \sigma_{ap}(A) \subset \sigma(A)$

All points lying outside the circle of radius $\|A\|$ with center at the origin are regular. All points of the spectrum are in the disk;

$|\lambda| \leq \|A\|$.

The radius of the smallest disk with center at the origin containing the spectrum of the operator A is called spectral radius r_A of the operator A .

REFERENCES

1. Bollobás, B. (1990). *Linear Analysis: An Introductory Course*. Cambridge University Press.

2. Douglas, R. G., (1969). On the operator equation $S^*XT = X$ and related topics, *ActaSci. Maths (Szeged)*, 30, 19-32.
3. Halmos, P. R. (1967). *A Hilbert Problem Book, Second Edition*. Springer.
4. Limaye, B. V. (1981). *Functional Analysis*. J. Wiley.
5. Shah, N. C., & Sheth, I. H. (1975). Some result on quasi-hyponormal operators. *J, Indian Math*, 39, 255-259.