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THE LIE SYMMETRY ANALYSIS OF THIRD ORDER KORTEWEG-DE VRIES EQUATION

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Abstract
This study sought to analyse the Lie symmetry of third order Korteweg-de Vries equation. Solving nonlinear partial differential equations is of great importance in the world of dynamics. Korteweg-de Vries equations are partial differential equations arising from the theory of long waves, modelling of shallow water waves, fluid mechanics, plasma fluids and many other nonlinear physical systems, and their effects are relevant in real life. In this study, Lie symmetry analysis is demonstrated in finding the symmetry solutions of the third-order KdV equation of the form$u_t + u_{xxx} + uu_x = 0$. The study systematically showed the formula to find the specific solution attained by developing prolongations, infinitesimal transformations and generators, adjoint symmetries, variation symmetries, invariant transformation and integrating factors to obtain all the lie groups presented by the equation. In conclusion, infinitesimal generators, group transformations and symmetry solutions of third-order KdV equation are acquired using a method of Lie symmetry analysis. This was achieved by generating infinitesimal generators, which act on the KdV equation to form infinitesimal transformations. It can be seen from the solutions of this paper that the Lie symmetry analysis method is an effective and best mathematical technique for studying linear and nonlinear PDEs and ODEs.

Key terms: KdV equation, Lie symmetry analysis, prolongations, symmetry solutions.

1.0 INTRODUCTION

Nonlinear Partial differential equations model many phenomena in mathematical, engineering and physical fields, which include fluid mechanics, nonlinear optics, plasma, optic fibres and any more. Obtaining their symmetry solutions plays a significant role in nonlinear models in physics and real life. Various scholars have investigated KdV equations of different orders using various methodologies to attain their solutions which involves; (Chavan & Pancha, 2014; Elzaki & Eman 2012; Hemeda, 2012) applied homotopy perturbation method, Wazzan (2016) used extended hyperbolic function method, (Malfliet, 1992) applied the tanh- function method, Elzaki (2012) used new integral transform, Wazwaz (2007) applied tanh-coth method, Wazwaz (2002) used the sine-cosine method, (Hirota, 2004) used the Hirota bilinear formalism, Ablowitz and Clarkson, (1991) applied the inverse scattering method and many more techniques.

A third order KdV is considered given as (Brauer, 2000; Riseborough, 2011)

$$u_t + u_{xxx} + uu_x = 0 \qquad (1.1)$$

which is used widely in the field of physics and arises in the theory of long waves and modelling shallow water waves and many other fields. In this article, the Lie symmetry analysis method is applied to determine the symmetry solutions of the third-order KdV equation, which has yet to be considered. This equation arises in long waves in shallow water theory and other physical systems.

2.0 LITERATURE REVIEW

Alvaro (2012) presented the use of the sn-ns technique to find the solutions of nonlinear partial differential equations. They showed that the well-known tanh-coth technique is a specific case of the sn-ns technique. It was found that it was the most suitable of all methods studied since it provided elliptic function results as well as trigonometric and hyperbolic solutions. In the cases when the sn-ns technique did not work, other methodologies, such as the tanh-coth method, were used.

Amna and Syed (2012) applied Tanh-Coth Method to obtain travelling wave results of nonlinear differential equations. The method was considered since it was entirely compatible with the complexity of the systems and was believed to be highly effective and could handle nonlinear systems of useful physical nature. The tanh-coth technique was effectively used to find solitary wave results, and its presentation was consistent and effective and gave additional results. The applied technique would be used in further works to determine more new results for other types of nonlinear wave equations. Such results are rational solutions, polynomial solutions, and travelling wave solutions that have been obtained by many authors under diverse methods. Though their results were soliton and periodic solutions, they had many potential applications in physics.

Ahmet and Adem (2011) established abundant travelling wave results for nonlinear-coupled evolution equation. The technique was used to obtain travelling wave solutions and solitons solutions of nonlinear-coupled evolution equation. The tanh–coth method combined with the Riccati equation presents a wider applicability for handling nonlinear wave equations. Alvaro and Cesar (2009) considered the general projective Riccati equation technique and the Exp-function technique that were used to construct generalised soliton results and periodic results to a special KdV equation containing both variable coefficients and forcing terms. The techniques certainly worked well for a large class of exciting nonlinear equations. The major advantage of those techniques was their ability to significantly reduce the scope of computational work compared to current techniques.

Qu and Wang (2007) gave a group of asymmetric difference systems to solve the Korteweg-de Vries equation. Such systems were the full explicit difference scheme and the fully implicit scheme. They constructed an alternating segment explicit-implicit difference outline for solving the KdV equation. The system was linear and unconditionally stable by the analysis of the linearisation formula and could be used directly on the parallel computer. The numerical experiments demonstrated that the technique had high precision.

Marchant (2004) considered solitary wave interaction for a higher-order Korteweg–de Vries equation. The equation was obtained by retaining third-order terms in the perturbation expansion, whereby first-order terms are reserved. Since the third-order coefficients fulfilled the algebraic relationship, the third-order KdV equation was transformed into the KdV equation, thus deriving a two-soliton solution by means of the transformation. The phase shift corrections were found by the fact that the collision was asymptotically elastic. Both elastic and inelastic collisions were considered. In addition, phase shift corrections were found numerically for a series of solitary wave amplitudes. It was found that mass was not conserved by the third-order KdV equation but changes during the interaction of the solitary waves.

Shah et al. (2020) presented a fractional view of third-order Kortewege-De Vries equations using a refined analytical method termed the Mohand decomposition technique. This was done by using Caputo fractional derivative operator to obtain fractional derivatives in the considered problems. Some numerical models were presented to demonstrate the efficacy of the technique for both integer and fractional order problems. It was explored that the proposed technique had a similar rate of convergence when compared to the homotopy perturbation transform scheme. The resulting graphs established the best agreement with the exact results of the problems and showed that if the sequence of fractional orders approaches the integer order, then the fractional order results of the problems converge to an integer order solution. The technique was easy to implement and thus could be used to solve similar nonlinear fractional order partial differential equations.

3.0 METHODOLOGY

Lie Symmetry Analysis

The symmetry groups of transformations are given as

$$x^* = X(x,t,u;\varepsilon),\ t^* = T(x,t,u;\varepsilon),\ u^* = U(x,t,u;\varepsilon) \quad (1.2)$$

in addition, they contain corresponding infinitesimals of the form

$$\mu(x,t,u) = \frac{\partial X(x,t,u;\varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0},\ \alpha(x,t,u) = \frac{\partial T(x,t,u;\varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0},\ \beta(x,t,u) = \frac{\partial U(x,t,u;\varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0} \quad (1.3)$$

Let the generator G, of (1.1) be given in the form of

$$G = \mu(x,t,u)\frac{\partial}{\partial x} + \alpha(x,t,u)\frac{\partial}{\partial t} + \beta(x,t,u)\frac{\partial}{\partial u} \quad (1.4)$$

In addition, the coefficient functions μ, α β are found so that the corresponding one-parameter Lie group of transformations $x^* = X(x,t,u;\varepsilon),\ t^* = T(x,t,u;\varepsilon),\ u^* = U(x,t,u;\varepsilon)$ results to a symmetry group of (1.1).

Since equation (1.1) is a third-order partial differential equation, we use the third prolongation, which is given as

$$G^{[3]} = \alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \lambda(x,t,u)\frac{\partial}{\partial u} + \lambda^{t}\frac{\partial}{\partial u_t} + \lambda^{x}\frac{\partial}{\partial u_x} + \lambda^{tt}\frac{\partial}{\partial u_{tt}} + \lambda^{tx}\frac{\partial}{\partial u_{tx}} + \lambda^{xx}\frac{\partial}{\partial u_{xx}}$$

$$+\lambda^{ttt}\frac{\partial}{\partial u_{ttt}} + \lambda^{ttx}\frac{\partial}{\partial u_{ttx}} + \lambda^{txx}\frac{\partial}{\partial u_{txx}} + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}}$$

When $G^{[3]}$ acts on equation (1.1), we acquire

$$G^{[3]}\left[u_t + u_{xxx} + uu_x\right] = 0 \qquad (1.5)$$

β_x

Therefore, equation (1.1) becomes

$$\left[\mu(x,t,u)\frac{\partial}{\partial x} + \alpha(x,t,u)\frac{\partial}{\partial t} + \beta(x,t,u)\frac{\partial}{\partial u} + \beta^{x}\frac{\partial}{\partial u_x} + \beta^{t}\frac{\partial}{\partial u_t} + \beta^{xx}\frac{\partial}{\partial u_{xx}} + \beta^{xt}\frac{\partial}{\partial u_{xt}} + \beta^{tt}\frac{\partial}{\partial u_{tt}} + \beta^{xxx}\frac{\partial}{\partial u_{xxx}} + \beta^{xtx}\frac{\partial}{\partial u_{xtx}} + \beta^{xxt}\frac{\partial}{\partial u_{xxt}} + \beta^{xtt}\frac{\partial}{\partial u_{xtt}} + \beta^{ttt}\frac{\partial}{\partial u_{ttt}}\right][u_t + u_{xxx} + uu_x] = 0 \qquad (1.6)$$

Upon simplifying and differentiating partially with respect to the partial variables $u_t\ u_x\ u_{xx}, u_{tt}, u_{xt}, u_{xxt}, u_{xxx}, u_{xtx}, u_{xtt}, u_{ttt}$, and x,t,u as algebraic variables, we obtain the infinitesimal of the form

$$\beta^{t} + \beta^{xxx} + \beta u_x + \beta^{x} u = 0 \qquad (1.7)$$

which need to be satisfied, ensuring that $u_t = -u_{xxx} - uu_x$ whenever it appears in the equation.

Upon replacing u_t by $-u_{xxx} - uu_x$ whenever it appears in the equation, we find several monomials in the first, second and third-order partial derivatives u.

Thus the determining equations formed are:

Monomial terms	Equation	
u_{xxt}:	$\alpha_x + \alpha_u u_x = 0$	(i)
$u^2{}_{xxx}$:	$\alpha_u = 0$	(ii)
$u_{xx}{}^2$:	$-3\mu_u = 0$	(iii)
$u_x u_{xxt}$:	$\alpha_u = 0$	(iv)
$u_x u_{xxx}$:	$\mu_u = 0$	(v)
u_{xx}:	$3\mu_{xx} - 3\beta_{ux} = 0$	(vi)
$u_x u_{xx}$:	$\mu_u + 3\beta_{uu} - 6\mu_{ux} - 9\mu_{ux} = 0$	(vii)
$u_x{}^2$:	$3\beta_{uux} = 0$	(viii)
u_x:	$\beta - \mu_t + [\beta_u - \mu_x]u + 3\beta_{xuu} = 0$	(ix)
1:	$\beta_{xxx} + u\beta_x + \beta_t = 0$	(x)

The solutions of equations (i)-(x) yield the infinitesimals μ, α, β given as

$$\mu = c_1 + c_3 t + c_4 x \qquad (1.8a)$$

$$\alpha = c_2 + 3c_4 t \qquad (1.8b)$$

$$\beta = c_3 + (-2c_4 u) \qquad (1.8c)$$

We write μ, α, β in the standard basis form represented as

$\frac{w_1}{\downarrow}$ $\frac{w_1}{\downarrow}$ $\frac{w_1}{\downarrow}$ $\frac{w_1}{\downarrow}$

$$\left.\begin{aligned} \mu &= 1.c_1 + 0.c_2 + t.c_3 + 1.c_4 x &= c_1 + c_3 t + c_4 x \\ \alpha &= 0.c_1 + 1.c_2 + 0.c_3 + 3.c_4 t &= c_2 + 3c_4 t \\ \beta &= 0.c_1 + 0.c_2 + 1.c_3 - 2.c_4 u &= c_3 + (-2c_4 u) \end{aligned}\right\} \quad (1.9)$$

Then we formulate the equivalent Lie algebra of the basis generators $w_1,\ w_2,\ w_3,\ w_4$ in (1.9) of the form $w_i = \widehat{\mu_i}\frac{\partial}{\partial x} + \widehat{\alpha_i}\frac{\partial}{\partial t} + \widehat{\beta_i}\frac{\partial}{\partial u}$: $\widehat{\mu_i},\ \widehat{\alpha_i},\ \widehat{\beta_i}$ are the coefficients c_i in the standard results of $\mu,\ \alpha,\ \beta$. Thus, the $w_i,\ \ i = 1, 2, 3, 4$ are acquired from the presentation in equation (1.9) as given below

$$w_1 = \frac{\partial}{\partial x},\ \ w_2 = \frac{\partial}{\partial t},\ \ w_3 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial x},\ \ w_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} \quad (1.10)$$

The Lie groups admitted by equation (1.1) are determined by solving the conforming Lie equations, which yields groups as presented below.

We now use $w_1,\ w_2,\ w_3, and\ w_4$ to find the solution for each R_i.

Therefore

$$w_1 = \frac{\partial}{\partial x};\ R_1{:}X(x,t,u;\varepsilon) \to \quad X_1(x+\varepsilon,t,u) \quad (1.11a)$$

$$w_2 = \frac{\partial}{\partial t};\ R_2{:}X(x,t,u;\varepsilon) \to \quad X_2(x+\varepsilon,t,u) \quad (1.11b)$$

$$w_3 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial x};\ R_3{:}\ X(x,t,u;\varepsilon) \to \quad X_3(x+\varepsilon t,t,u+\varepsilon) \quad (1.11c)$$

$$w_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u};\ R_4{:}\ X(x,t,u;\varepsilon) \to \quad X_4(e^{\varepsilon}x + e^{\varepsilon}\,t, e^{-2\varepsilon}u) \quad (1.11d)$$

Equations (1.11a) represent space translation, (1.11b) – time translation, (1.11c) – Galilean boost and (1.11d) scaling.

Table 1: Commutator Table for the Groups

	w_1	w_2	w_3	w_4
w_1	0	0	0	w_1
w_2	0	0	w_1	$3w_2$
w_3	0	$-w_1$	0	$-2w_3$
w_4	$-w_1$	$-3w_2$	$2w_3$	0

By exponentiation, it shows that if $u = f(x,t)$ is a solution of equation (1.1), so are

$u^{(1)} = f(x-\varepsilon,t)$,

$u^{(2)} = f(x,t-\varepsilon)$,

$u^{(3)} = f(x-\varepsilon t,t) + \varepsilon$,

$u^{(4)} = e^{-2\varepsilon}f(e^{-\varepsilon}x, e^{-3\varepsilon}t)$.

for all $\varepsilon \in \mathbb{R}$.

Group Invariant Solutions and the general solution

We base our solutions on the symmetry group solutions obtained initially for the particular group invariant solutions.

(i) Travelling Wave Solutions

The group, in this case, is the same as the translational group whereby when the invariants $y = x - ct,\ \ v = u$ are used; the following reduced equation is obtained as

$v_{yyy} + vv_y - mv_y = 0$

Upon integration, we obtain

$$v_{yy} + \frac{1}{2}v^2 - mv = c$$

Multiplying by v_y and integrating for the second time, we obtain

$$\frac{1}{2}v_y^2 = -\frac{1}{6}v^3 + \frac{1}{2}mv^2 + cv + h$$

where c and h are arbitrary constants. Thus the general solution is given in elliptic form as $u = \wp(x - ct + \delta)$, δ is a phase shift.

If $u \to 0$ as $|x| \to \infty$, then $c = h = 0$, we obtain an accurate solution

$v = 3msech^2\left[\frac{1}{2}\sqrt{my} + \delta\right]$, given that m is positive. Thus, the soliton solution to the KdV equation is given as

$$u(x,t) = 3msech^2\left[\frac{1}{2}\sqrt{m}(x - mt) + \delta\right].$$

(ii) Galilean Invariant Solutions

These are generated by $\frac{\partial}{\partial u} + t\frac{\partial}{\partial x}$. Given that $t > 0$, $y = t$, $v = tu - x$ are independent invariants used to calculate

$u = y^{-1}(x + v)$, $u_x = y^{-1}$, $u_{xxx} = 0$, $u_t = y^{-2}(yv_y - v - x)$ x is the arbitrary variable

Thus, the reduced equation is $\frac{dv}{dy} = 0$; thus, the solution becomes

$u = (x + \delta)/t$ where δ is an arbitrary constant.

If we add a time translational component to this group, we obtain a more interesting Galilean invariance solution given by

$$u(x,t) = k\left(x - \frac{1}{2}bt^2\right) + bt.$$

(iii) Scale-Invariant Solutions

The groups of scaling symmetries are given as

$(x,t,u) \to (\beta x, \beta^3 t, \beta^{-2}u)$

When> 0, the invariants are $y = t^{\frac{-1}{3}}x$, $v = t^{2/3}u$.

Thus, we find $u_x = t^{-1}v_y$, $u_{xxx} = t^{-5/3}v_{yyy}$, $u_t = -\frac{1}{3}t^{-5/3}(yv_y + 2v)$

giving rise to a reduced equation as

$$v_{yyy} + vv_y - \frac{1}{3}yv_y - \frac{2}{3}v = 0$$

Upon integration, it gives rise to the equation

$$w_{yy} = \frac{1}{18}w^3 + \frac{1}{3}yw + c$$

4.0 RESULTS AND DISCUSSION

Since we have obtained the invariant solutions and the symmetry solutions in the previous sections, the space translation invariant solutions are all constant. Therefore, they appear trivially among the other solutions.

5.0 CONCLUSION

In this article, infinitesimal generators, group transformations and symmetry solutions of third-order KdV equation are acquired using a method of Lie symmetry analysis. This was achieved by generating infinitesimal generators, which act on the KdV equation to form infinitesimal transformations. It can be seen from the solutions of this paper that the Lie symmetry analysis method is an effective and best mathematical technique for studying linear and nonlinear PDEs and ODEs.

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