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Optimisation as an Algebraic Axiom in Cartesian Closed Categories: Applications to Dataflow Networks

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Abstract

The aim of this paper is to present a framework that integrates optimisation into algebraic structures in Cartesian Closed Categories (CCCs). Traditional mathematical methods treated optimisation as an external process, which limited its foundational role in mathematics. Inspired by the finite nature hypothesis and Hilbert's sixth problem, which calls for the axiomatisation of physical principles, this study formalises optimisation as an intrinsic algebraic axiom in a way that aligns with Hilbert's vision of uniting mathematical and physical laws. The framework builds on Lawvere's categorical treatment of metric spaces and Birkhoff's HSP theorem, which the study uses to define an optimisation algebra. The study then provides proof that the class of such algebras forms a variety in universal algebra and demonstrates categorical soundness within CCCs. The proposed approach guarantees that optimisation is inherent within algebraic systems, establishing natural substructures of optimal elements and facilitating compositional reasoning in computational models. Applying this framework in dataflow networks demonstrates convergence to optimal steady states, enhancing resource utilisation and system efficiency. Future research includes using the framework in enriched categories, distributed systems and incorporating the operator in tools that can be used to solve real-world problems.

Key terms: Cartesian closed category, monoid, optimisation algebra, variety.

1.0 INTRODUCTION
This paper addresses a significant gap in the foundations of mathematics, wherein optimisation is typically regarded as an external process rather than an intrinsic component of algebraic systems. This gap has limited the ability to accurately and consistently model computing processes in accordance with algorithmic principles. The finite nature hypothesis in physics posits that nature is finite and digital, and that everything in nature must have a digital information representation (Corry, 2018). Hilbert's 6th problem on the principle of least action further motivates extending mathematical foundations to include optimisation principles (Corry, 2018). Embedding optimisation as an algebraic axiom provides a unified framework which enables compositional reasoning and expands applicability in computation.

A key concept in this study is optimisation algebra, which is an algebraic structure (more precisely, a monoid) enhanced with an operator that chooses optimal elements while maintaining the underlying operations. A monoid is a set that can facilitate consecutive composition and accumulation due to the presence of an identity element and an associative binary operation (Burris & Sankappanavar, 1981).

Extending this notion, a variety provides a methodical approach to studying families of optimisation-enhanced structures by representing a class of algebras characterised by shared identities (Birkhoff, 1935). These ideas inevitably result in Cartesian Closed Categories (CCCs), which are categories with finite products and an internal hom-functor satisfying the exponential law. CCCs enable the integration of optimisation principles into higher-order structures (Lawvere, 1973; Mac Lane, 1998).

This study presents a formal framework for embedding optimisation in Cartesian closed categories, summarising recent developments in categorical semantics (Garner & Kock, 2025) alongside essential principles in category theory and universal algebra (Kelly, 1982; Burris & Sankappanavar, 1981).

2.0 LITERATURE REVIEW
According to classical works by Kelly (1982) and Boyd and Vandenberghe (2004), optimisation was seen as something that happened outside of algebraic structures. It was primarily applied within calculus and convex optimisation. Category theory gives us a natural framework through cartesian closed categories, where optimisation can be directly integrated into algebraic systems. This is because category theory offers a suitable alternative that aligns with the computational and compositional reasoning found in present-day systems. Even so, optimisation itself has not yet been formalised as an algebraic operator inside CCCs, despite the fact that much research has included computational operations or effects into algebraic or categorical settings. There is currently no study that clearly specifies a monoid-based optimisation algebra with axioms and shows that this class of structures integrates categorically into CCCs and generates a variety under Birkhoff's HSP theorem.

The potential for internalising optimisation in categorical frameworks was raised by Lawvere, who showed that metric spaces could be described as categories enriched over the extended real line (Lawvere, 1973). Litvinov and Maslov strengthened Lawvere's work by substituting min/max for addition, resulting in a semiring structure that naturally supports optimisation (Litvinov & Maslov, 2005). Birkhoff's HSP theorem gives us the algebraic tools we need to show that these kinds of optimisation structures make up a variety (Birkhoff, 1935). To do this, you need to list a set of identities and show that they are closed under homomorphic images, subalgebras, and direct products. This gives them the same structural strength as other universal algebra varieties. Universal algebra unifies the shared characteristics of various algebraic structures into a cohesive framework, facilitating the examination of concepts such as homomorphisms and direct products in a broader context (Burris & Sankappanavar, 1981; Wechler, 2012). These findings have been used in previous studies to show that optimisation-augmented monoids indeed fit into universal algebra categories, but none of them have addressed optimisation as a native operator in category semantics or explicitly framed this integration in CCCs.

Kelly's (1982) seminal work formalised closed categories and, together with Mac Lane's (1998) work, established an algebraic flexibility framework that is crucial for optimisation integration. Recent research has also connected algebraic theories with computational processes. Garner and Kock (2025) enhance universal algebra through enriched multicategories within duoidal categories, offering new methods for modelling computations involving effects. This contemporary viewpoint demonstrates that operations like minimisation or optimisation can be incorporated as fundamental actions within categorical frameworks, further solidifying the integration of optimisation concepts.

Shiebler et al. (2021) extended gradient descent and Newton's technique categorically by developing generalised optimisation theory employing categorical constructs such as Cartesian reverse derivatives. In their assessment of category-theoretic frameworks in machine learning, Jia et al. (2025) highlighted three approaches that embed computational activities categorically: gradient-based, probabilistic, and topos-theoretic. Koenig (2022) modelled computational side-effects as free algebras in enriched categorical settings, establishing a connection between algebraic effects and game semantics.

The body of work surrounding operator-augmented algebras adds further insight, considering its lengthy history with computation. For example, averaging operators in algebras have been investigated as endomorphisms f that fulfil the condition $f(xf(y)) = f(x)f(y)$ (Zhao & Bai, 2014). Such operators, including the optimisation operator ϕ, maintain structural integrity while also identifying specific elements. Their examination highlights how the enhancement of an algebra with an additional operator leads to the formation of natural substructures that are invariant under that operator—mirroring ϕ's function in optimisation algebras. Stepanov's early work on reduction operators in commutative semigroups showed how associative and commutative laws naturally support parallelisation in functional programming (Stepanov, 1981). This computational viewpoint fits in well with the existing framework, where optimisation has real-world implications for concurrency and stability in dataflow networks, and also strengthens algebraic foundations.

The reviewed literature shows a clear trend: optimisation has evolved from an external analytical procedure to a notion that is increasingly incorporated into algebraic and categorical frameworks. Lawvere, Kelly, and Birkhoff's foundational works established the theoretical framework for structural integration, while universal algebra supplied the formal tools for defining varieties. There is increasing interest in categorical semantics for computing and optimisation, as seen by recent studies ranging from operator-augmented algebras to enriched multicategories and algebraic effects. These studies do not, however, treat optimisation as a first-class algebraic axiom in Cartesian Closed Categories. This study formalises optimisation algebras and demonstrates their categorical soundness to address the necessity for a cohesive framework, shown by the convergence of historical foundations and modern advancements.

3.0 METHODOLOGY

The study introduces a new definition of an optimisation algebra. This new definition is inspired by principles of a category and the universal algebra conditions. An optimisation algebra is defined by using a monoid $(A,\bullet,e)$ which is equipped with an operator $\varphi : A \to A$ satisfying the following axioms:

(i) Operation-preservation $(\varphi(x)\bullet\varphi(y) = \varphi(x\bullet y)$

(ii) Idempotency: $\varphi(\varphi(x)) = \varphi(x)$ for all $x \in A$

(iii) Identity preservation: $\varphi(e) = e$

These axioms make sure that the operator φ selects 'optimal' representatives while preserving the algebraic structure. The paper employs algebraic proof methods based on category theory and universal algebra to demonstrate the soundness of this framework. In particular, it shows that the operator maintains monoid operations by using homomorphism arguments and structural induction to confirm closure features.

The study then provides proof using Birkhoff's HSP theorem that the class of such algebras forms a variety, hence providing a vigorous algebraic setting for optimisation. Birkhoff's HSP theorem (1935) states that a class of algebras defined by a set of identities is closed under Homomorphic images, Subalgebras, and direct Products, and therefore forms a variety. This formalisation is done within the Cartesian Closed Categories framework, making it possible to merge algebraic, topological and computational reasoning. Finally, the study provides additional proofs to show that the image of φ is a natural submonoid of optimal elements on which φ acts as the identity, reinforcing the operator's categorical and algebraic soundness.

4.0 FINDINGS AND DISCUSSION

Definition: Monoid

A monoid $(A,\bullet,e)$ is a set A equipped with an associative binary operation $\bullet$ and an identity element e.

Proposition 1 *— Monoid axioms*

Let $(A,\bullet,e)$ be a set A with a binary operation $\bullet$ and an element $e \in A$. Suppose $\bullet$ associative and e is a two-sided identity. Then $(A,\bullet,e))$ is a monoid.

Proof

By hypothesis;

(i) $x, y \in A, x\bullet y \in A$ (this is assumed as $\bullet$ is a binary operation on A).

(ii) Associativity. For all, $x, y, z \in A,\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
(iii) Identity. $e \cdot x = x$ and $x \cdot e = x$

These three properties are exactly the definition of a monoid. ■

Definition: Optimisation Algebra

In universal-algebraic terms, the identities satisfied are;

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(ii) $e \cdot x = x = x \cdot e$
(iii) $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$
(iv) $\varphi(\varphi(x) = \varphi(x)$
(v) $\varphi(e) = e$

The study defines an optimization algebra as a monoid equipped with an optimization operator $\varphi : A \to A$ satisfying:

(i) Operation-preservation: $(\varphi(x) \cdot \varphi(y)) = \varphi(x \cdot y)$
(ii) Idempotency: $\varphi(\varphi(x)) = \varphi(x)$ for all $x \in A$
(iii) Identity preservation: $\varphi(e) = e$

The tuple$(A, \bullet, e, \varphi)$ then defines an optimisation algebra.

Proposition 2— $\mathrm{Im}(\varphi)$ ***is a submonoid of A***

Let $(A, \bullet, e, \varphi)$ be an optimization algebra and write $\mathrm{Im}(\varphi) = \{\varphi(x) \mid x \in A\}$. Then $\mathrm{Im}(\varphi)$ is a submonoid of A; i.e., it is closed under ,con the identity e, and its operation is associative.

Proof

(i) Closure
Take arbitrary $a, b \in \mathrm{Im}(\varphi)$. By definition there exist $x, y \in A$ with $a = \varphi(x)$ and $b = \varphi(y)$. Then using operation-preservatior $a \bullet b = \varphi(x) \bullet \varphi(y) = \varphi(x \bullet y) \in im(\varphi)$.

(ii) Identity
By identity preservation $\varphi(e) = e$, so $e \in im(\varphi)$

(iii) Associativity.
The operation $\bullet$ on $im(\varphi)$ is associative because it is the restriction of $\bullet$ on A.

The figure below indicates the operation preservation of φ .

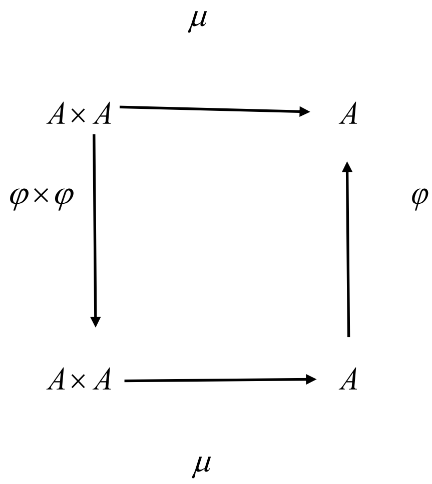

Figure 1: Operation preservation

Where;

(i) The top arrow shows the multiplication map $\mu(x \cdot y) = x \cdot y$

(ii) Left arrow indicates $\varphi \times \varphi$

(iii) The bottom arrow shows μ

(iv) Right side arrow show φ

Commutativity of this square is $(\varphi(x) \cdot \varphi(y)) = \varphi(x \cdot y)$.
Therefore $\mathrm{Im}(\varphi)$ is a submonoid. ■

Proposition 3- ***φ acts as identity on its image.***

In an optimization algebra $(A, \bullet, e, \varphi)$ for every $a \in im(\varphi)$ $\varphi(a) = a$. Equivalently, φ restricted to $im(\varphi)$ is the identity map.

Proof

Let $a \in im(\varphi)$. Then $a = \varphi(x)$ for some $x \in A$. Using idempotency,
$\varphi(a) = \varphi(\varphi(x)) = \varphi(x) = a$.
Thus φ fixes every element of its image. ■

This idempotency can be illustrated with below, showing that φ restricts to the identity map on its image: $\varphi(\varphi(x)) = \varphi(x)$.

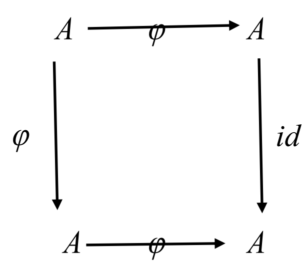

Figure 2: **Idempotency of** φ

Proposition 4 - φ ***is a monoid homomorphism onto its image.***

If φ satisfies operation-preservation and $\varphi(e)=e$, then $\varphi:(A,\bullet,e)\rightarrow(im(\varphi),\bullet,e)$ is a surjective monoid homomorphism.

Proof

(i) Homomorphism property
For all $x,y\in A$, $\varphi(x\bullet y)=\varphi(x)\bullet\varphi(y)$, so φ preserves the operation.

(i) Identity preservation
$\varphi(e)=e$

(ii) Surjectivity

By definition, φ maps onto $im(\varphi)$.

Hence φ is a surjective homomorphism. ∎

Lemma- The class of optimization algebras is a variety.

Let O be the class of all structures $(A,\bullet,e,\varphi)$ satisfying the identities:

(i) $(x\cdot y)\cdot z=x\cdot(y\cdot z)$

(ii) $e\cdot x=x\cdot e=x$

(iii) $\varphi(x\cdot y)=\varphi(x)\cdot\varphi(y)$

(iv) $\varphi(\varphi(x)=\varphi(x)$

(v) $\varphi(e)=e$

By Birkhoff's HSP theorem, any class defined by a set of identities is a variety; hence O is closed under homomorphic images, subalgebras, and direct products.

Application to Dataflow Networks

Modelling parallel and concurrent computation requires dataflow networks, especially those based on FIFO channels. Feedback loops can be demonstrated to converge to a single optimal steady state by integrating optimisation into the categorical model of these networks. This facilitates accurate resource utilisation analysis and increases system stability and efficiency.

5.0 CONCLUSION AND RECOMMENDATIONS

Conclusion: The idea of optimisation algebras is used in this paper to formalise optimisation as a basic algebraic axiom within Cartesian Closed Categories. The paper demonstrated that the optimisation operator defines a submonoid of optimal elements and that these structures form a variety. The framework's theoretical elegance and practical usefulness are demonstrated through its application to dataflow networks.

Recommendations: In the future, this method will be extended to other algebraic structures, and computational applications will be further investigated.

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