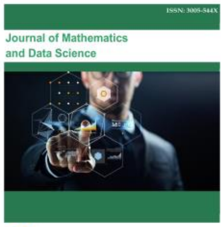

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ON CONDITION OF QUASISIMILAR, QUASINORMAL AND PURE DOMINANTOPERATORS IN EXISTENCE OF ESSENTIAL SPECTRA IN AN INFINITE DIMENSIONAL HILBERT SPACES

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Abstract

The problem of finding conditions of quasisimilar, pure dominant operators in connection with related essential spectrum has been considered by several authors. In this paper, we show that quasisimilar pure dominant operators have their essential spectra equal to their spectra, provided one of the interfering quasiaffinities is compact. We will consider T as a pure dominant operator, K as a compact operator having dense range and let $KT = TK$ so that we can investigate the conditions of the spectrum of T and essential spectrum of T. In this study an effort will be made to give relevant examples to illustrate conditions of pure parts and hence deduce results of equality of essential spectra.

Key terms: Dunford's property (c), pure dominant, pure parts, quasiaffinities, quasisimilar

1.0 INTRODUCTION

The previous studies in operator theory shows that for any operator A, the essential spectrum of A is contained in the spectrum of A. In this paper we show that quasisimilar pure dominant operators and study the essential spectra. An operator $A \in B(H)$ is said to be a quasiaffinity if A is both one-one and has dense range. Two operators A and B are said to be similar if there is an invertible operator S such that $AS = SB$, while A and B are said to be quasisimilar if there exist quasiaffinities X and Y such that $AX = XB$ and $BY = YA$. It should be noted that an operator $T \in H$ is said to satisfy Dunford's property (c) if for each closed subset F of the complex plane the corresponding local spectrum sub-space $H_T(F) - \{x \in H: \sigma(T, x) \subset F\}$ is closed. In this study, we also make the effort to show that quasisimilar pure dominant operators have their essential spectra equal to their spectra provided one of the interfering quasiaffinities is compact.

2.0 NOTATION AND TERMINOLOGY

In this paper, let H denote an infinite dimensional separable Hilbert space and $\boldsymbol{B(H)}$**:** algebra of bounded linear operators on H. If $T \in B(H)$ write $N(T)$ and $R(T)$ for the null space and range of T: $\rho\ (T)$ for the resolvent set of T; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the eigenvalues of T; $\pi_{0f}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0i}(T)$ for the eigenvalues of infinite multiplicity. $V_{\mathcal{H}}$ will denote unilateral shift on $\mathcal{H}$. It is familiar that if $T \in B(H)$ then T is regular if and only if T has closed range. An operator $T \in B(H)$ is called *upper semi-Fredholm* if it has closed range with finite-dimensional null space and *lower semi-Fredholm* if it has closed range with its range of finite co-dimension. If T is either upper or lower upper or lower Semi-Fredholm, we call it Semi-Fredholm and if T is both upper and *lower semi-Fredholm*, we call it Fredholm. An operator $T \in B(H)$ is called *Weyl* if it is Fredholm of index zero. An operator $T \in B(H)$is called Browder if it is Fredholm "of finite ascent and descent": The essential spectrum $\sigma_e(T)$, and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined by;

$\sigma_e(T) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not Fredholm}\};$

$\sigma_b(T) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not Browder}\}:$

then

$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) \cup \text{acc}\ \sigma(T) \subseteq \eta\, \sigma_e(T),$

3.0 RESULTS AND DISCUSSION

Theorem 1

Suppose that T is a pure dominant operator, K is a compact operator having dense range and $KT = TK$. Then spectrum of T is equal to essential spectrum of T.

Remark 1

It is at this point that we also pick up the quest of delving into this theory. It is important to state the following definition to introduce the Dunford's property (c).

Definition 1

Let $x \in \mathcal{H}$ We define $\varrho_T(x)$ to be the set of complex numbers α for which there exists a neighbourhood V_α of α with μ analytic on having values in H such that $(zI - T)u(z) = x$ on $\forall_\alpha$.

We say that T has the single valued extension property (in short SVEP) if $(zI - T)\, u(z) = 0$ implies $u = 0$ for any analytic function u defined on any donation D of a complex plane with values in H.

An operator $T \in H$ is said to satisfy Dunford's property (c) if for each closed subset F of the complex plane the corresponding local spectrum subspace $H_T(F) - \{x \in H : \sigma(T, x) \subset F\}$ is closed.

Theorem 2

Suppose A and B are dominant operators satisfying Dunford's property (c) and are quasisimilar with at least one of the implementing quasiaffinities compact, then we have A and B are equal spectra and also equal essential spectra.

Remark 2

Note that the problem of looking for conditions under which the essential spectrum is equal to spectrum of a given operator has also been considered by a number of authors. In particular, Williams (1980) managed to show that there are several cases under which quasisimilar operators A and B have equal essential spectra also proved the following result on equality of spectrum and essential spectrum for a given operator.

Lemma 1

(i) If A is compact then so is A^*

(ii) If A is compact and B is bounded then AB and BA are also compact.

Remark 3

We not from the Lemma above that if A is compact and B is bounded then their products are also compact and we now state the following theorem if we take compact quasiaffinities.

Theorem 3

Let $A, B \in B(H)$ be quasisimilar pure dominant operators with at least one of the intertwining quasiaffinities compact. Then we have:

$\sigma_e(A) = \sigma(A)$ *and* $\sigma_e(B) = \sigma(B)$.

Proof:

Since A and B are quasisimilar there exist two quasiaffinities X and Y such that

$AX = XB$ and $BY = YA$ .This condition is well known and very clear.

We also have that either X or Y is compact implies XY and $YX(commmutants)$ are compact operators each with dense range. It can also be verified easily that $[A, XY] = 0$ and$[B, YX] = 0$. Now from theorem1 above hence the result we have $\sigma_e(A) = \sigma(A)$ and $\sigma_e(B) = \sigma(B)$

Q.E.D.

Corollary 1

Let $A, B \in B(H)$ be quasiinvertible operators with either A and B compact. If AB and BA are pure dominant operators then we have $\sigma_e(AB) = \sigma(AB)$ and $\sigma_e(BA) = \sigma(BA)$.

Proof

From the condition of quasi similar operators, we deduce the result of pure dominant operators and hence essential condition. $\sigma_e(AB) = \sigma(AB)$. **Q.E.D**

Theorem 4

Let A be a pure dominant operator and B be such that $AX = XB$ implies $A^*X = XB^*$ where X is a compact quasiaffinity, then $\sigma_e(A) = \sigma(A)$.

Proof

Since $AX = XB$ implies $A^*X = XB^*$ it can easily be verified that

$[A, XX^*] = 0$ and

$[B, X^*X] = 0$

Where XX^* is compact with dense range and hence its clear; $\sigma_e(A) = \sigma(A)$. **Q.E.D.**

Corollary 2

Let A and B be quasisimilar pure dominant operators, which satisfy Dunford's condition (c) with at least one of the implementing quasiaffinities compact. Then we have $\sigma_e(AB) = \sigma(A) = \sigma(B) = \sigma_e(B)$

Remark 4

Let now consider T_1 and T_2 be quasisimilar hyponormal operators on infinite dimensional Hilbert spaces.

Clary (1975) proved in [2] that $\sigma(T_1) = \sigma(T_2)$. Duggal (1996) showed that there are several cases which imply $\sigma_e(T_1) = \sigma_e(T_2)$. For example, if T_1 and T_2 are both biquasisimilar, if T_1 and T_2 are both weighted shifts (bilateral or unilateral), or if T_1 and T_2 are both partial isometries, then $\sigma_e(T_1) = \sigma_e(T_2)$.

The purpose here is to prove that if T_1 and T_2 are both quasinormal, then $\sigma_e(T_1) = \sigma_e(T_2)$. Suppose that T is an operator. Thus, in order to prove that two quasisimilar quasinormal operators T_1 and T_2 have equal essential spectra, it suffices to study the pure parts of T_1 and T_2. Hence we shall begin by considering the pure parts of quasinormal operators. Denote the index of T. It is well-known that $\sigma_e(T) = \{\lambda \epsilon\, C : T - \lambda \text{ is not Fredholm}\}$. A *hole* in $\sigma_e(T)$ is a bounded component of $C/\sigma_e(T)$. It is also well-known that if H is a hole is $\sigma_e(T)$, then $i(T - \lambda)$ is constant on H. We shall first prove the following Theorem.

Theorem 5

Suppose that T_1 and T_2 are quasisimilar quasinormal operators an infinite dimensional Hilbert spaces then $\sigma_e(T_1) = \sigma_e(T_2)$.

Proof

Note T_1 is unitarily equivalent to $N_i \oplus V_{\mathcal{H}_i} \hat{P}_i$ on $\mathcal{H}_i + \widehat{\mathcal{H}}_i$ where N_i is a normal operator on the Hilbert space $\mathcal{H}_i$ and P_i is a positive definite operator on $\mathcal{H}_i, (i = 1,2)$ implies that both $T_1$and T_2 are normal. Thus, in this case, T_1 and T_2 are unitarily equivalent and $\sigma_e(T_1) = \sigma_e(T_2)$. Hence we may assume that both $\mathcal{H}_1$ and $\mathcal{H}_2$ are nonzero. In order to complete the proof, it suffices to show that $\sigma_e(V_{\mathcal{H}_1}\hat{P}_1) = \sigma_e(V_{\mathcal{H}_2}\hat{P}_2)$. There exist quasiaffinites X and Y such that $X(N_1 \oplus V_{\mathcal{H}_1}\hat{P}_1) = (N_2 \oplus V_{\mathcal{H}_2}\hat{P}_2)X$ and $(N_1 \oplus V_{\mathcal{H}_1}\hat{P}_1) = Y(N_2 \oplus V_{\mathcal{H}_1}\hat{P}_2)$

Example 1

Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let V be the unilateral shift on $\mathcal{H}$ defined by $Ve_n = e_{n+1}, n = 1, 2, \ldots..$ then there exist a quasiaffinity W and a positive operator R in $\mathcal{L}(\mathcal{H})$ by $Ue_n = \left({}^1/_2{}^n\right)_{e_n,}$ $n = 1, 2, \ldots ...$ The operators U is a quasiaffinity and $1/2VU = UV$.Let$T_1 = \hat{V} \oplus 1/2 \oplus 1/2\hat{R}$ on $\widehat{\mathcal{H}} \oplus \mathcal{H} \oplus \widehat{\mathcal{H}} \to \widehat{\mathcal{H}} \oplus \widehat{\mathcal{H}}$

By$X\left((x_1, x_2, x_3, \ldots) \oplus x_0 \oplus (y_1, y_2, y_3, \ldots.)\right) = (x_1, x_2, x_3, \ldots) \oplus (Wx_0, x_1, x_2, x_3, \ldots)$ and

$Y: \widehat{\mathcal{H}} \oplus \widehat{\mathcal{H}} \to \widehat{\mathcal{H}} \oplus \mathcal{H} \oplus \widehat{\mathcal{H}}$ b $Y\left((x_1, x_2, x_3, \ldots) \oplus (y_1, y_2, y_3, \ldots)\right) = (x_2, x_3, \ldots) \oplus Ux_1 \oplus (y_1, y_2, y_3, \ldots)$

It is clear that X and Y are quasiaffinites and a routine calculation shows that $XT_1 = T_2X$ and $T_1Y = YT_2$. Hence T_1 and T_2 are quasisimilar. Note that the pure part of T_1 is $\hat{V} \oplus 1/2V$ and the pure part of T_2 is $\hat{V}$. We shall show now that $\hat{V} \oplus 1/2V$ and $\hat{V}$ are not quasisimilar by using the same argument. Suppose that there exists a quasiaffinity $Z: \widehat{\mathcal{H}} \oplus \mathcal{H} \to \widehat{\mathcal{H}}$ such that $Z(\hat{V} \oplus 1/2V) = \hat{V}Z$. Define $W: \mathcal{H} \to \widehat{\mathcal{H}}$ by $Wx = Z(0 \oplus x)$.

Let $m \to \mathcal{H}$ is injective. Since $\hat{V}$ is completely nonunitary, $\hat{V}|m$ is a nonunitary isometry thus for $1/2 < |\lambda| < 1, \lambda$ is an eigenvalues of $(\hat{V}|m)^*$ and thus also of $(1/2)V^*$. The last statement is clearly a contradiction. Therefore, the pure parts of T_1 and T_2 are not quasisimilar J. Conway also proved that subnormal operators are similar if and only if their normal parts are unitarily equivalent and their pure parts are similar. Thus, the equality of the essential spectra of quasisimilar quasinormal operators is not a result of similarity.

Remark 5

The following example shows that two quasisimilar quasinormal operators need not be similar even if both operators are pure.

Example 2

Let $\mathcal{H}$ be a Hilbert space with a orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $c_1 = 1, d_1 = 1/2, d_{2n} = c_{2n} = 1/4,\ n = 1, 2, \ldots.,$ and $d_{2n+1} = c_{2n+1} = 1, n = 1, 2, \ldots.$ Define positive definite operators P_1 and P_2 on $\mathcal{H}$ by $P_1e_n = c_ne_n$ and $P_2e_n = d_ne_n, n = 1,2, \ldots$ Define an operator Y_1 on $\mathcal{H}$ by the following: let $Y_1e_1 = e_2$ and for each positive integer n let $Y_1e_{2n} = e_{2n+2}$ and $Y_1e_{2n+1} = e_{2n-1}$. Let $X_n = P_2^n(P_1^{-1})^n$ and $Y_{n+1} = P_1^nY_1(P_2^{-1})^n,\ n = 1,2, \ldots$ Observe that, for each positive integer $3n$, we have $\|X_n\| = \|Y_n\| = 1, X_{n+1}P_1 = P_2X_n,$ and $Y_{n+1}P_2 = P_1Y_n$.

Let $X = \sum_{n=1}^{\infty} \oplus X_n$ and $X = \sum_{n=1}^{\infty} \oplus Y_n$. Then X and Y are quasiaffinities on $\widehat{\mathcal{H}}$, $XV_{\mathcal{H}}\hat{P}_1 = V_{\mathcal{H}}\hat{P}_2X$, and $V_{\mathcal{H}}\hat{P}_1Y = YV_{\mathcal{H}}\hat{P}_2$. Hence $T_1 = V_{\mathcal{H}}\hat{P}_1$ and $T_2 = V_{\mathcal{H}}\hat{P}_2$ are quasinormal operators. We show next that T_1 andT_2 are not similar. The operator T_1 is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus c_nV$ and T_2 is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus d_nV$. It follows that $\|(T_1 - 1/2)x\| \geq 1/4\|x\|$ for each x in $\widehat{\mathcal{H}}$. Thus $T_1 - 1/2$ has closed range. Since $1/2\,V$ is one of the direct summands of $\sum_{n=1}^{\infty} \oplus d_nV$ and $1/2\,V - 1/2$ does not have closed range, if follows that $T_1 - 1/2$ does not have closed range. Hence T_1 and T_2 are not similar.

4.0 CONCLUSION

It should be noted that together with the results discussed above, Conway (1981) proved in that the normal parts of quasisimilar subnormal operators are unitarily equivalent. In that paper he also provided an example1, which showed that the pure parts of quasisimilar subnormal operators need not be quasisimilar. Close scrutiny of his example will reveal that one of the two quasisimilar subnormal operators is not quasinormal. However, a slight modification of his example will show that the parts of quasisimilar quasinormal operators need not be quasisimilar. Therefore, the pure parts of T_1 and T_2 are not quasisimilar J. Conway (1981) also proved that subnormal operators are similar if and only if their normal parts are unitarily equivalent and their pure parts are similar. Hence, the two quasisimilar quasinormal operators are not similar by example 2. Thus, the equality of the essential spectra of quasisimilar quasinormal operators is not a result of similarity.

5.0 RECOMMEDATIONS

In this last section, we shall briefly present some areas, which are of interest for possible future Study. The study of Browder spectrum its properties, will draw similarities of essential spectrum and Browder spectrum. Also, slight modification of the example given by Conway (1981), will show that the parts of quasisimilar, quasinormal operators need not necessarily be quasisimilar.

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